Macaulay Coefficients and Decomposing Lex Segments

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Abstract

Let $S = k[x_1, x_2, \ldots, x_n]$ and M be a graded module over S. Lex ideals allow us to give a bound, in terms of binomial coefficients, for the rate at which the Hilbert function H_M of M grows by decomposing them into direct sums of monomial spaces. This is normally done when M is a quotient of S by an ideal, but we claim a similar process can be done when M is an ideal. These two processes have interesting dual properties, namely our main result that the induced Macaulay coefficients of a degree-δ monomial in n variables form a set partition of $\{0, 1, \ldots, n + \delta - 2\}$.

Macaulay's Theorem

Macaulay's theorem shows why lex ideals are of interest in the study of Hilbert functions.

Theorem. (Macaulay) Let $\mathcal{I} = \bigoplus_i I_i$ be a graded ideal of S. Then there exists a lex ideal $\mathcal{L} = \bigoplus_i L_i$ of S such that $H_{\mathcal{L}} = H_{\mathcal{I}}$. In particular, if $S_1 = (x_1, \ldots, x_n)$, we have the following bounds for all i:

$$\dim S_1 L_i \leq \dim I_{i+1} \qquad \dim S/(S_1 L_i) \geq \dim (S/\mathcal{I})_{i+1}.$$

Macaulay Representations and Coefficients

We refer to a standard combinatorial result.

Theorem. Let s and p be positive integers. Then there exists a unique decreasing sequence of non-negative integers $s_p > s_{p-1} > \cdots > s_1$ such that

$$s = \binom{s_p}{p} + \binom{s_{p-1}}{p-1} + \cdots + \binom{s_1}{1}. \qquad (1)$$

The expression (1) is called the *p^{th} Macaulay representation of s*, and the integers $s_p, s_{p-1}, \ldots, s_1$ are called the *p^{th} Macaulay coefficients of s*.

Computing the Dimension of a Lex Quotient

The quotient $S/\mathcal{L}$ by a lex ideal $\mathcal{L}$ may be written as a direct sum of a monomial space and another lex quotient of smaller degree, treating each as a k-vector space over bases of monomials. For example, in $k[a, b, c, d]$:

$$\mathcal{Q}^4(b^2cd) = b\mathcal{Q}^3(bcd) \oplus (c, d)^4,$$

where $\mathcal{Q}^4(b^2cd)$ is the space spanned by all degree-4 monomials strictly lex-smaller than b^2cd and $\mathcal{Q}^3(bcd)$ is the space spanned by all degree-3 monomials strictly lex-smaller than bcd. We may repeat the decomposition for $\mathcal{Q}^3(bcd)$ and continue to obtain a full decomposition of the lex quotient $\mathcal{Q}^4(b^2cd)$:

$$\mathcal{Q}^4(b^2cd) = (c, d)^4 \oplus b(c, d)^3 \oplus b^2(d)^2 \oplus b^2c(0)^1.$$

The dimension is thus

$$\dim \mathcal{Q}^4(b^2cd) = \binom{5}{4} + \binom{4}{3} + \binom{2}{2} + \binom{0}{1} = 10.$$

Computing the Dimension of a Lex Ideal

The decomposition for lex ideals is less-studied than that of lex quotients. A lex ideal may be written as a direct sum of a monomial space and another lex ideal with fewer variables, treating each as a k-vector space over bases of monomials. For example, in $k[a, b, c, d]$:

$$\mathcal{I}_4(b^2cd) = a^1(a, b, c, d)^3 \oplus a^0\mathcal{I}_3(b^2cd),$$

now considering the monomials strictly lex-larger than b^2cd. The full decomposition is

$$\mathcal{I}_4(b^2cd) = a(a, b, c, d)^3 \oplus b^3(b, c, d)^1 \oplus b^2c^2(c, d)^0.$$

The dimension is thus

$$\dim \mathcal{I}_4(b^2cd) = \binom{6}{3} + \binom{3}{2} + \binom{1}{1} = 24.$$

Formulas for the Induced Macaulay Coefficients

The dimensions of $\mathcal{Q}^4(b^2cd)$ and $\mathcal{I}_4(b^2cd)$ are naturally written in their Macaulay representations via these decompositions. Thus, it would be much nicer to determine their Macaulay coefficients directly from the monomial in question rather than from the corresponding decompositions.

Definition. Let $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a degree-δ monomial in $k[x_1, \ldots, x_n]$. For $0 \leq i \leq n-1$, define the *i^{th} coarse tail of m* to be $\mathrm{ct}_i(m) := x_{i+1}^{\alpha_{i+1}} x_{i+2}^{\alpha_{i+2}} \cdots x_n^{\alpha_n}$. If we write $m = x_{j_1} x_{j_2} \cdots x_{j_\delta}$ where j_p is nondecreasing, define the *i^{th} fine tail of m* to be $\mathrm{ft}_i(m) := x_{j_{i+1}} x_{j_{i+2}} \cdots x_{j_\delta}$ for $0 \leq i \leq \delta - 1$.

Theorem. Let $\mathcal{I}_n(m)$ ($\mathcal{Q}^\delta(m)$) be the lex space spanned by the degree-δ monomials in n variables that are lex-larger (-smaller) than m. Let s_i (t_i) be the $(n-1)^{\text{th}}$ (δ^{th}) Macaulay coefficients of its dimension. Then

$$s_i = i + \deg(\mathrm{ct}_{n-i}(m)) - 1 \qquad \text{and} \qquad t_i = n - \min(\mathrm{ft}_{\delta-i}(m)) + i - 1$$

for each i.

We will call the integers s_i the *n^{th} ideal coefficients of m* and the integers t_i the *δ^{th} quotient coefficients of m*.

Example

Let $m = b^2cd$ in $k[a, b, c, d]$. Then we have

$$\begin{aligned} s_1 &= 1 + \deg(d) - 1 = 1 & t_1 &= 4 - \min(d) + 1 - 1 = 0 \\ s_2 &= 2 + \deg(cd) - 1 = 3 & t_2 &= 4 - \min(cd) + 2 - 1 = 2 \\ s_3 &= 3 + \deg(b^2cd) - 1 = 6 & t_3 &= 4 - \min(bcd) + 3 - 1 = 4 \\ & & t_4 &= 4 - \min(b^2cd) + 4 - 1 = 5, \end{aligned}$$

which matches our earlier computations. Observe that the 3^{rd} ideal coefficients of b^2cd are not only disjoint from its 4^{th} quotient coefficients, but together they partition the set $\{0, 1, 2, 3, 4, 5, 6\}$. This property holds in general.

Main Result

Theorem. Let $m \in k[x_1, \ldots, x_n]$ be a monomial of degree δ. Let $\mathcal{S}, \mathcal{T}$ be the sets of n^{th} ideal and δ^{th} quotient coefficients of m, respectively. Then $\{\mathcal{S}, \mathcal{T}\}$ forms a set partition of $\{0, 1, \ldots, n + \delta - 2\}$.