How to prove that $K^{-1}K = I$

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The classical inverse Kostka matrix

- $h_k = \sum_{1\le i_1\le\cdots\le i_k} x_{i_1}\cdots x_{i_k}$ $\quad h_\lambda = h_{\lambda_1}\cdots h_{\lambda_\ell}$
- **Jacobi–Trudi formula** $s_\lambda = \det(h_{\lambda_i+j-i})_{i,j=1}^{\ell(\lambda)}$
- The **inverse Kostka matrix** K^{-1} is the transition matrix from $\{s_\lambda\}_{\lambda\vdash n}$ to $\{h_\lambda \vdash n\}$ in **Sym**.
- **Eğecioğlu and Remmel**: fill partition diagrams with special rim hooks and assign signed weights to these fillings to obtain the inverse Kostka matrix entries.

Problem (Eğecioğlu and Remmel '90)

Provide a combinatorial proof that $K^{-1}K = I$.

Into the world of NSym

- $\{H_1, H_2, \dots\}$ =algebraically independent functions that *don't commute.* For any composition $\alpha = (\alpha_1, \dots, \alpha_\ell)$
$$H_\alpha = H_{\alpha_1}H_{\alpha_2}\cdots H_{\alpha_\ell}$$
NSym is generated by $H_1, H_2, \dots$ (with no relations)
- **Immaculate functions** (Berg et al 14')
$$\mathfrak{S}_\alpha = \mathfrak{det}(H_{\alpha_i+j-i})$$
where $\mathfrak{det}$ is the **NSym** determinant.
- **Immaculate tableau** filling of composition diagram with weakly increasing rows and strictly increasing first column.
- **NSym Kostka matrix** $\tilde{K}_{\alpha,\beta}$=number of immaculate tableau of shape α and content β.

Main results

We solve Eğecioğlu's and Remmel's problem by first solving the analogous problem in NSym and then using this to solve the original problem.

Tunnel hook coverings (Allen–Mason '23)

Let $\alpha = (\alpha_1, \dots, \alpha_\ell)$ be a composition. Draw this diagram in English notation. A **tunnel hook covering** of α is a disjoint collection $T = (\tau_1, \dots, \tau_\ell)$ of **tunnel hooks** such that

1. There is a tunnel hook τ_i starting in each row $i \in [\ell]$ and proceeding down and to the left.
2. τ_i starts in column
$$\begin{cases} \alpha_i & \text{if the row is not totally covered by } \tau_1,\dots,\tau_{i-1} \\ \alpha_i+1 & \text{if the row is exactly covered by } \tau_1,\dots,\tau_{i-1} \\ \alpha_i+2k & \tau_1,\dots,\tau_{i-1} \text{ cover } k \text{ cells in row } i \text{ outside of } \alpha \end{cases}$$

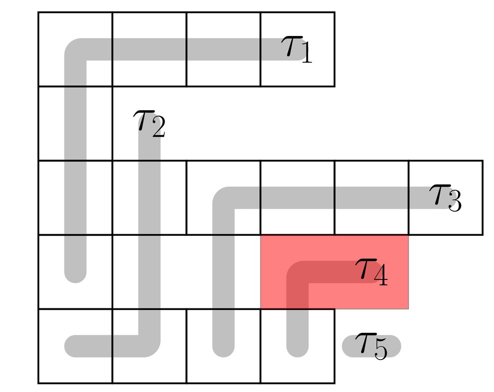
- **Weight** $\Delta(\tau_i)$ =#{cells in τ_i} $-$#{cells in row i outside of α covered by some τ_j}
- **Content** $\alpha(T) = flat((\Delta(\tau_1), \dots, \Delta(\tau_\ell))$ where *flat* removes any 0's
- **Sign** $\mathrm{sgn}(\tau_i) = (-1)^{j-i}$ if τ_i ends on row j. $\mathrm{sgn}(T) = \prod_{i=1}^{\ell}\mathrm{sgn}(\tau_i)$.

Theorem (Allen–Mason 2023)

Let $\mathrm{THC}_{\beta,\alpha}$ be the set of THC's of content β and shape α. For $\alpha, \beta \vdash n$, we have
$$\tilde{K}^{-1}_{\beta,\alpha} = \sum_{T\in \mathrm{THC}_{\beta,\alpha}} \mathrm{sgn}(T).$$

Note: Allen–Mason extend this to all integer sequences.

Example ($\alpha = (4,1,6,1,4)$)



$\alpha(T) = (7,4,6,-1)$. $\mathrm{sgn}(T) = (-1)^{3+3+2+1+0} = -1$.

NSym Eğecioğlu and Remmel problems

Provide **sign-reversing involutions** to establish the following identities

$$\delta_{\alpha,\beta} = (\tilde{K}\tilde{K}^{-1})_{\alpha,\beta} = \sum_{(S,T)} \mathrm{sgn}(T)$$

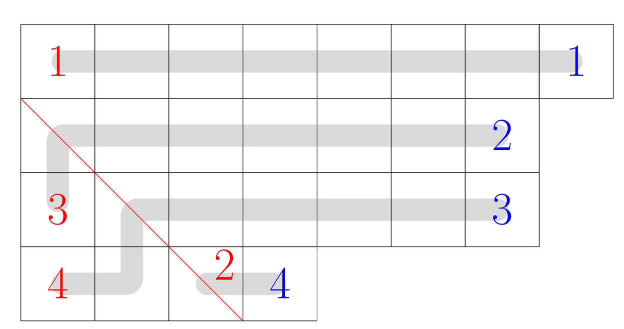
where the sum is over all (S,T) of
- S=immaculate tableau shape α
- T=tunnel hook covering of shape β
- S and T have **same content**

$$\delta_{\alpha,\beta} = (\tilde{K}^{-1}\tilde{K})_{\alpha,\beta} = \sum_{(T,S)} \mathrm{sgn}(T)$$

where the sum is over all (T,S) of
- T=tunnel hook covering of content α
- S=immaculate tableau content β
- S and T have **same shape**

Note: We focus on $\tilde{K}^{-1}\tilde{K} = I$ for this poster. See full paper for $\tilde{K}\tilde{K}^{-1} = I$.

Permutations and tunnel hook coverings

$T =$ 

The *permutation* $\mathrm{perm}(T)$ of a THC T is defined by $\mathrm{perm}(T)_i = j$ if τ_i ends on diagonal j (starting with main diagonal and going down).

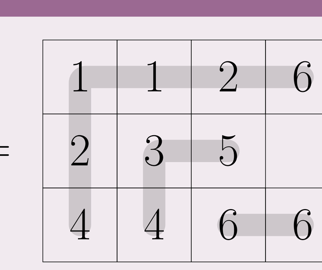
$$\mathrm{perm}(T) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$$

If $\beta = (\beta_1, \dots, \beta_\ell)$, $T \mapsto \mathrm{perm}(T)$ is a bijection $\bigsqcup_\alpha \mathrm{THC}_{\alpha,\beta} \to \mathfrak{S}_\ell$ such that $\mathrm{sgn}(\mathrm{perm}(T)) = \mathrm{sgn}(T)$

Involution proving $\tilde{K}^{-1}\tilde{K} = I$ combinatorially (Allen–C.–Mason 2025+)

Let $\sigma = \mathrm{perm}(T)$.

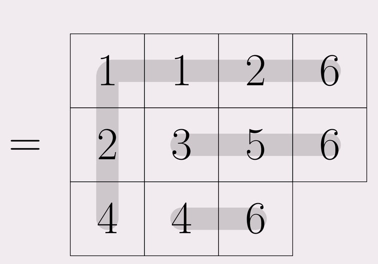
1. Let $m = \max(S)$; $\mathcal{R} = \{\sigma_r \mid m \text{ appears in row } r \text{ of } S\}$. Select r such that $\sigma_r = \min(\mathcal{R})$.
2. If $\sigma_r = r$ and S has only m's in row r, then remove final row, induct, reattach.
3. Suppose either $\sigma_r = r$ and the final row is not all m's or $\sigma_r \neq r$.
   1. Set $U(c) = S(c)$ for all cells c except for the last cell of row r. Append an m to the end of row $p = \sigma^{-1}(q+1)$, where $q = \sigma_r$.
   2. Let V be the THC of shape $sh(U)$ with $\mathrm{perm}(V) = s_q\sigma$.

$(S,T) =$ 

$\mathrm{perm}(T) = \sigma = 321$

$m = 6$, $\mathcal{R} = \{1,3\}$

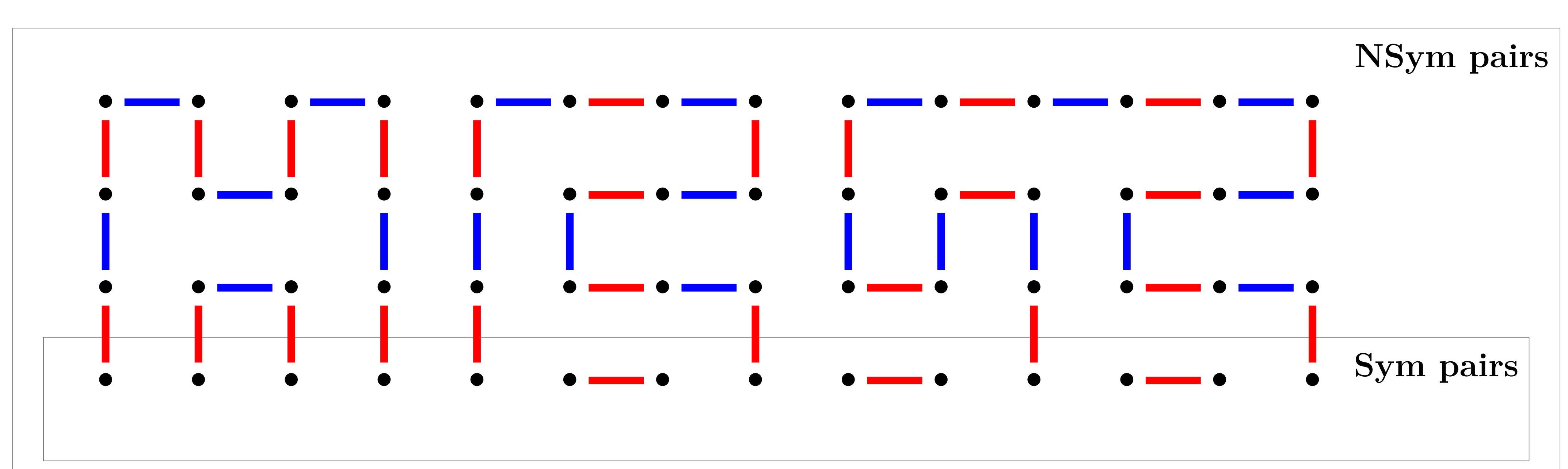
$r = 3$, $q = \sigma_r = 1$

$(U,V) =$ 

$p = \sigma^{-1}(q+1) = 2$

$\mathrm{perm}(V) = s_1 \cdot 321 = 312$

Sym involution proving $K^{-1}K$ combinatorially (Allen–C.–Mason (2025+)



Sym involution (explanation)

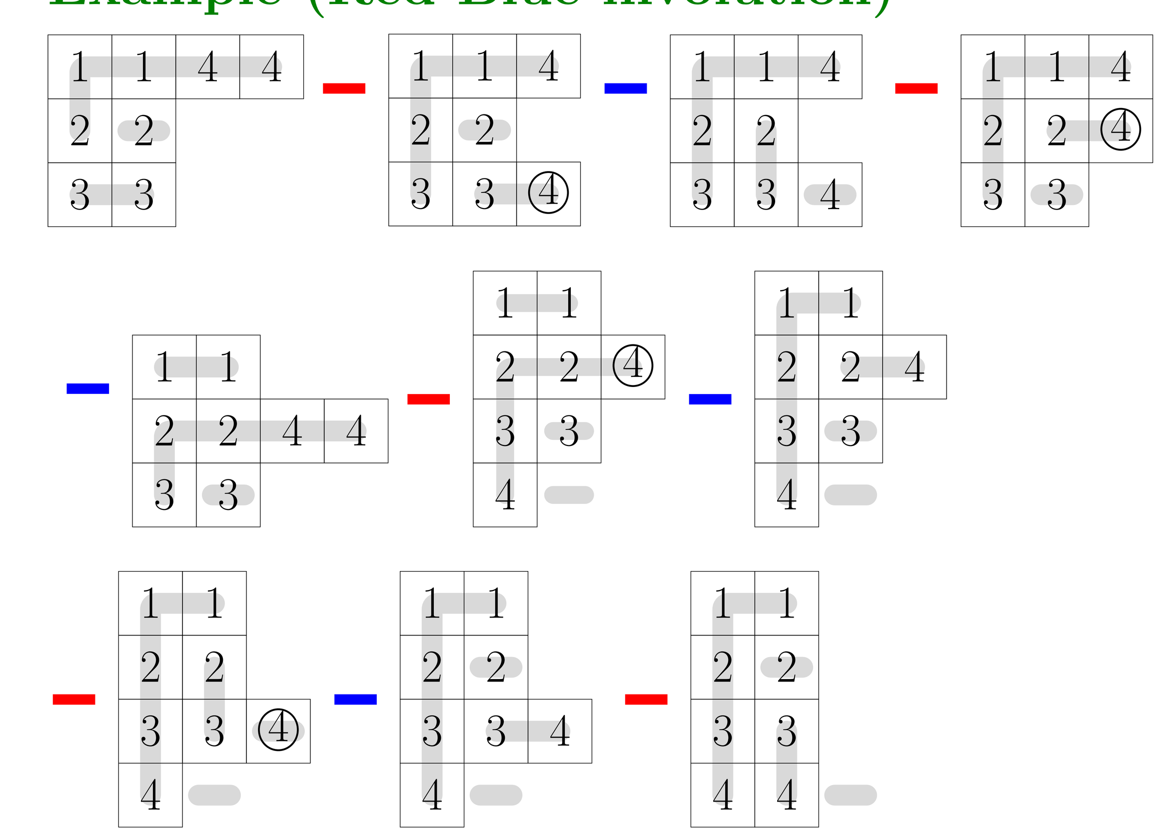
Goal: Provide a sign reversing involution proving
$$\delta_{\lambda,\mu} = (K^{-1}K)_{\lambda,\mu} = \sum_{(T,S)} \mathrm{sgn}(T)$$
where the sum is over all (T,S) of
- T = THC of content α that rearranges to λ
- S = semistandandard Young tableau of content μ
- S and T have **same shape**

The involution is comprised of two parts:
- **Red**: Our **NSym** involution
- **Blue**: *Gasharov-style* involution on **NSym pairs** (T,S) that are not **Sym pairs**, meaning that S is an immaculate tableau that is NOT a SSYT.

We prove that
1. **Red** and **Blue** are distinct sign–reversing involutions
2. **Blue** never produces a **Sym pair**

Therefore, the set of **NSym pairs** decomposes into a set of **Red**–**Blue** paths that are all of *odd length* with each endpoint a **Sym pair**. Since this is the composition of an odd number of sign-reversing maps, it is sign-reversing. □

Example (Red-Blue involution)



Further directions

- Understand the relationship between our involution and that of Sagan–Lee ('06) for $(K^{-1}K)_{\lambda,1^n} = \delta_{\lambda,1^n}$
- Construct **Red**-**Blue** involutions for other tableau-like objects.

For Further Information

- Allen, E. and Mason, S. A combinatorial interpretation of the noncommutative inverse Kostka matrix. *arXiv:2207.05903 (2023)*
- E. E. Allen, K. Celano, S. K. Mason. *Proof of an inverse Kostka matrix problem posed by Eğecioğlu and Remmel and related identities in Sym and NSym.* 2025+. In preparation.
- Berg, C., Bergeron, N., Saliola, F., Serrano, L., and Zabrocki, M. A lift of the Schur and Hall-Littlewood bases to non-commutative symmetric functions. *Canad. J. Math.*, 66 (2014) 3:525-565.
- Eğecioğlu, Ö. and Remmel, J. A combinatorial interpretation of the inverse Kostka matrix. *Linear and Multilinear Algebra*, 26 (1990) 1-2:59-84.
- J. Lee, B.E. Sagan, *An algorithmic sign-reversing involution for special rim-hook tableaux*, J. Algorithms 59 (2) (2006) 149–161.