Geometric Vertex Decomposition, Substitution, and Toric Ideals of Graphs

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Vertex Decompositions of Simplicial Complexes

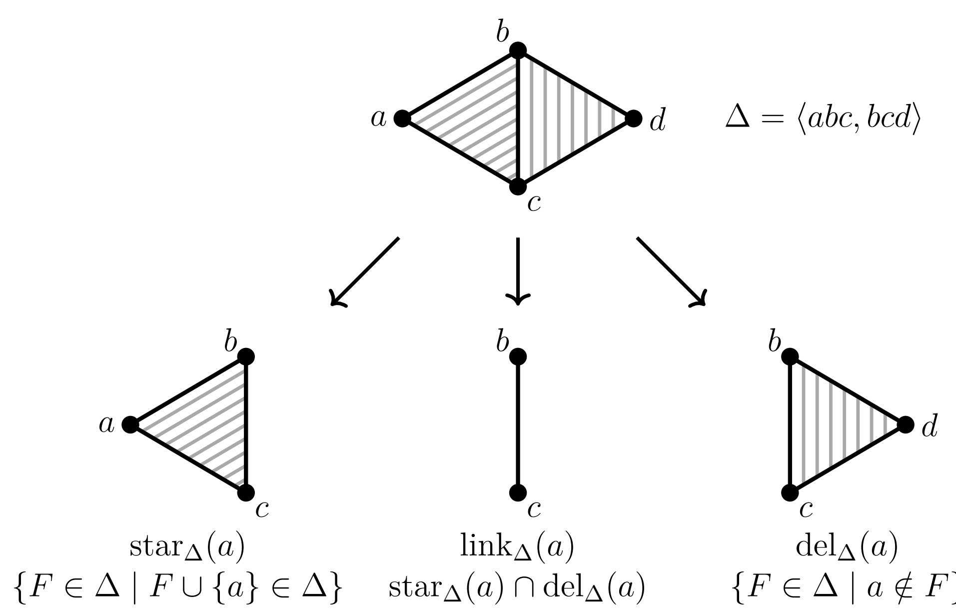

Figure 1. A **vertex decomposition** $\Delta = \mathrm{star}_\Delta(a) \cup \mathrm{del}_\Delta(a)$ of Δ.

Geometric Vertex Decomposition

vertex decompositions of simplicial complexes
$\Delta = \mathrm{star}_\Delta(v) \cup \mathrm{del}_\Delta(v)$

↕ Stanley–Reisner correspondence

vertex decompositions of squarefree monomial ideals
$I_\Delta = I_{\mathrm{star}_\Delta(v)} \cap I_{\mathrm{del}_\Delta(v)}$

↓ Knutson, Miller, Yong [7]

geometric vertex decompositions of polynomial ideals
$\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$

↓ Klein and Rajchgot [5]

geometric vertex decomposition *allowing substitution*
$\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y^d \rangle)$

If $I_\Delta = \langle v^{d_i} q_i \rangle$, where $d_i \in \{0,1\}$ and v does not divide any q_i, then,

$$I_{\mathrm{star}_\Delta(v)} = \langle q_i \rangle, \quad I_{\mathrm{link}_\Delta(v)} = I_{\mathrm{star}_\Delta(v)} + \langle v \rangle, \quad I_{\mathrm{del}_\Delta(v)} = \langle q_i \mid d_i = 0 \rangle + \langle v \rangle$$

Let y be a variable in $\mathbb{K}[x_1, \dots, x_n]$ and let $<$ be a y-*compatible* term order. Write $\mathcal{G} = \{y^{d_i} q_i + r_i\}_i$ a Gröbner basis for I with respect to $<$, where y and y^{d_i} do not divide any terms of any q_i and r_i, respectively. Then,

$$\mathrm{in}_y(I) = \langle y^{d_i} q_i \rangle, \quad C_{y,I} := \langle q_i \rangle, \quad N_{y,I} := \langle q_i \mid d_i = 0 \rangle.$$

Liaison Theory

Equidimensional schemes V_1 and V_2 with no common components are **G-linked** by $X := V_1 \cup V_2$ if X is Gorenstein.

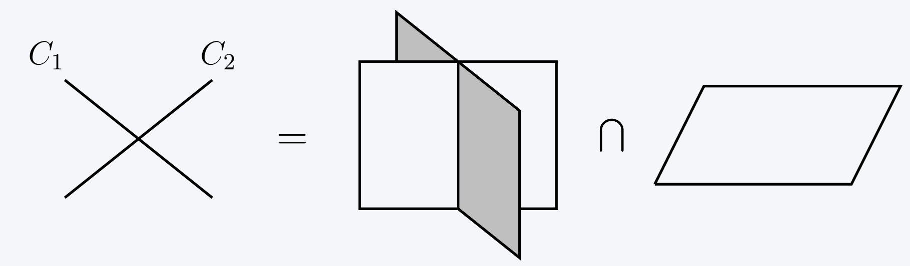

Figure 2. The intersection on the right is a G-link of C_1 and C_2 in $\mathbb{P}^3$ [8].

This resulting equivalence classes are called **Gorenstein liaison classes**.

Open Question ([6], 2001). Is every Cohen-Macaulay subscheme of $\mathbb{P}^n$ *in the* ***G****orenstein* ***li****aison* ***c****lass of a* ***c****omplete* ***i****ntersection (****glicci****)?*

Geometric Vertex Decomposition & Liaison

Key observation of Nagel and Römer, Klein and Rajchgot:

{geometric vertex decompositions} ⟷[4, 9] {elementary G-biliaisons of height 1}

This generalizes for geometric vertex decompositions *allowing substitution*:

{geometric vertex decompositions allowing substitution $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y^d \rangle)$} ⟷[1, 5] {elementary G-biliaisons of height d}

An unmixed ideal I is **geometrically vertex decomposable (allowing substitution)** [1, 5, 4] if it is unital or

- generated by indeterminates, or,
- admits a geometric vertex decomposition (allowing substitution)

$$\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y^d \rangle)$$

such that both $C_{y,I}$ and $N_{y,I}$ are geometrically vertex decomposable (allowing substitution).

Unital ideals and ideals generated by indeterminates are complete intersections. Hence, for homogeneous ideals:

geometrically vertex decomposable (allowing substitution) $\Longrightarrow$ *glicci* $\Longrightarrow$ *Cohen-Macaulay*

Geometric vertex decomposition allowing substitution also yields that for non-homogeneous ideals:

geometrically vertex decomposable $\Longrightarrow$ *Cohen-Macaulay*

Toric Ideals of Graphs

The **toric ideal** of a graph G is $I_G := \ker\left(\{v_i, v_j\} \in E \longmapsto v_i v_j\right)$. Generators of I_G correspond to closed even walks of G.

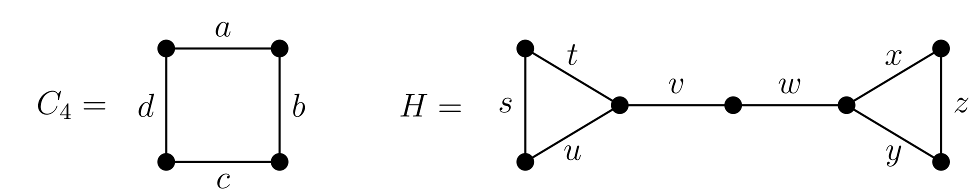

Figure 3. Toric ideals: $I_{C_4} = \langle ac - bd \rangle$ and $I_H = \langle sv^2xy - tuw^2z \rangle$.

Open Problem. Classify graphs whose toric ideals are Cohen-Macaulay.

- [2] Toric ideals of bipartite graphs are geometrically vertex decomposable, hence are Cohen-Macaulay
- [3] Gives a forbidden subgraph/odd-cycle condition that prevents Cohen-Macaulayness

GVD Allowing Substitution for Toric Ideals of Graphs

There are infinite families of graphs that are...

- GVD allowing substitution but not GVD
- weakly GVD allowing substitution but not weakly GVD

and hence are Cohen-Macaulay.

Idea: These families are closed under certain edge-gluing operations.

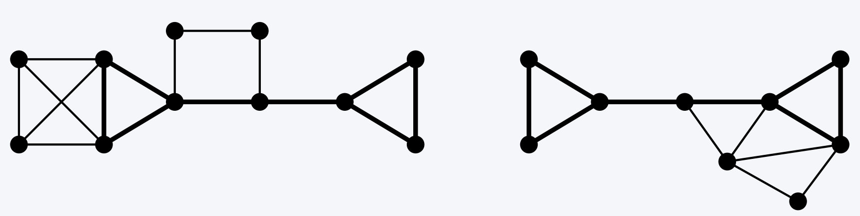

Figure 4. Graphs that are GVD allowing substitution but not GVD.

Acknowledgements & References

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