Real Matroid Schubert Varieties and Zonotopes

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Plan

Introduction

A *matroid Schubert variety* is a certain compactification of a vector space in $(\mathbb{P}^1)^n$, $n = 1, 2, 3, \ldots$

Introduction

Matroid Schubert varieties

1. were first defined by Ardila and Boocher [AB16];
2. are a "trendy" subject in combinatorial algebra and algebraic combinatorics, partly due to the successful resolution of Dowling and Wilson's *top-heavy conjecture* by Braden, Huh, Martherne, Proudfoot and Wang [BHM$^+$22], [BHM$^+$23] using the geometry of matroid Schubert varieties;
3. were rediscovered from the Poisson geometric perspective by Evens and Li [EL24], as a Poisson subvariety of *the variety of Lagrangian subalgebras of* $\mathfrak{g} \ltimes \mathfrak{g}^*$;
4. have representation theoretical significance, e.g. Ilin, Kamnitzer, Li, Przytycki and Rybnikov proved that they are intimately related to the moduli space of *"cactus flower curves"*, the *virtual cactus and symmetric groups* and *Gaudin subalgebras* [IKL$^+$24];
5. lead to an additive/tropical analogue of the theory of *toric varieties* [Cro23].

Definition

$\mathbb{F}$: any field, $A \in \mathrm{Mat}_{n\times d}(\mathbb{F})$: matrix of rank $d \rightsquigarrow$
Get embeddings

$$\mathbb{F}^d \xhookrightarrow{A} \mathbb{F}^n \hookrightarrow (\mathbb{P}^1(\mathbb{F}))^n \quad \rightsquigarrow$$

Will regard $\mathbb{F}^d$ as a locally closed subvariety of $(\mathbb{P}^1(\mathbb{F}))^n$ via the composition of the two embeddings.

Definition

The *matroid Schubert variety* Y *associated with* A is the closure of $\mathbb{F}^d$ in $(\mathbb{P}^1(\mathbb{F}))^n$.

Goal

Understand combinatorially the topology of Y in the case where $\mathbb{F} = \mathbb{R}$.

An example

Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$. $\rightsquigarrow$

The embeddings above are given by

$$\begin{array}{ccccc} \mathbb{F}^2 & \xhookrightarrow{A} & \mathbb{F}^3 & \xhookrightarrow{\qquad\qquad} & ((\mathbb{P}^1(\mathbb{F})))^3 \\ (a,b) & \longmapsto & (a,b,a+b) & \longmapsto & ([1:a],[1:b],[1:a+b]). \end{array} \quad \rightsquigarrow$$

If $([x_0 : x_1], [y_0 : y_1], [z_0 : z_1])$ are the homogeneous coordinates on $(\mathbb{P}^1(\mathbb{F}))^3$, then Y is cut out in $(\mathbb{P}^1(\mathbb{F}))^3$ by

$$x_1 y_0 z_0 + x_0 y_1 z_0 - x_0 y_0 z_1 = 0. \quad \rightsquigarrow$$

An example

Hence, Y has an affine paving given by

$$Y = \mathbb{F}^2 \sqcup \big(\mathbb{F} \times \{\infty\} \times \{\infty\}\big) \sqcup \big(\{\infty\} \times \mathbb{F} \times \{\infty\}\big) \sqcup \big(\{\infty\} \times \{\infty\} \times \mathbb{F}\big) \sqcup \{(\infty, \infty, \infty)\}.$$

The zonotope associated with A

From now on we assume that $\mathbb{F} = \mathbb{R}$.
Let $A_1, \ldots, A_n$ be the row vectors of the matrix A.

Definition

The zonotope Z associated with A is the Minkowski sum

$$Z := \sum_{i=1}^{n} [-A_i, A_i] = \left\{ \sum_{i=1}^{n} c_i A_i : c_i \in [-1, 1] \; \forall i \in [1, n] \right\}.$$

The zonotope associated with A

Z is a d-dimensional convex polytope sitting in $\mathbb{R}^d$, equipped with the Euclidean topology.
In particular, it makes sense to speak of two faces of Z being parallel.

Definition

Let $p, q \in Z$. We say that p is equivalent to q, $p \sim q$, if there exist faces $\mathcal{F}, \mathcal{G}$ of Z and a vector $x \in \mathbb{R}^d$ such that

$$p \in \mathcal{F}, \quad q \in \mathcal{G}, \quad \mathcal{F} + x = \mathcal{G} \quad \text{and} \quad p + x = q.$$

The set of equivalence classes in Z, equipped with the quotient of the Euclidean topology, will be denoted by $Z/{\sim}$.

Example, cont'd

Back to the example where $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$. $\rightsquigarrow$

The zonotope Z, as well as its parallel faces, is depicted as follows

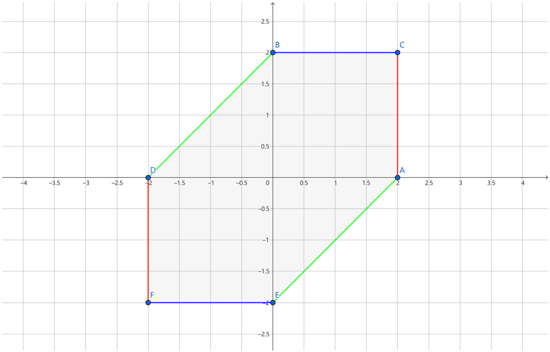

Figure: The zonotope for the Coxeter arrangement of type A_2

Example, cont'd

From the above it is clear that $Z/\sim$ is the 2-dimensional torus with two points identified.
Moreover, $Z/\sim$ has a CW complex structure (one cell for each equivalence class of parallel faces of Z) with

1. 1 cell of dimension 2;
2. 3 cells of dimension 1;
3. 1 cell of dimension 0.

Recall that the number of cells of dimensions $2, 1, 0$ in the affine paving of Y are also $1, 3, 1$!

The homeomorphism

$f : [-\infty, \infty] \to [-1, 1]$: any increasing homeomorphism. $\rightsquigarrow$
Get a map

$$\begin{aligned} \mathbb{R}^d &\longrightarrow \mathrm{Int}(Z) \\ x &\longmapsto \sum_{i=1}^{n} f(A_i x) A_i, \end{aligned}$$

where $\mathrm{Int}(Z)$ stands for the interior of Z.

The homeomorphism

Theorem (Jiang-L.)

There exists a unique continuous map $\varphi : Y \to Z/{\sim}$ making the diagram

$$\begin{array}{ccc} \mathbb{R}^d & \longrightarrow & \mathrm{Int}(Z) \\ \big\downarrow & & \big\downarrow \\ Y & \overset{\varphi}{\dashrightarrow} & Z/{\sim} \end{array}$$

commutative.
Moreover, the map $\varphi : Y \to Z/{\sim}$ is a homeomorphism and respects the CW complex structures.

Example, cont'd

To convince ourselves that the theorem is correct, let us consider again the example where $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$.

Take $f : [-\infty, \infty] \to [-1, 1],\ x \mapsto \frac{2}{\pi} \arctan(x)$. $\rightsquigarrow$
Get a homeomorphism $(\mathbb{P}^1(\mathbb{R}))^3 \to [-1, 1]^3/\text{parallel faces}$. $\rightsquigarrow$

Example, cont'd

The image of Y under this map is

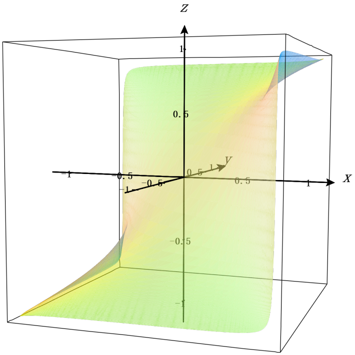

Figure: Image of Y

Flats of A

Definition

A *flat of A* is a subset $F \subseteq [1, n]$ which is maximal, with respect to inclusion, among all subsets $G \subseteq [1, n]$ that satisfy the condition

$$\mathrm{Span}\{A_i \colon i \in F\} = \mathrm{Span}\{A_i \colon i \in G\}.$$

The *rank* $\mathrm{rk}(F)$ of a flat F of A is

$$\mathrm{rk}(A) := \dim \mathrm{Span}\{A_i \colon i \in F\}.$$

The *join* $F \vee G$ of two flats F, G of A is the minimal, with respect to inclusion, flat of A that contains both F and G.

Fact

The flats of A are in natural bijection with the equivalence classes of parallel faces of Z.

Example, cont'd

The matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ has

1. 1 flat of rank 2: $\{1, 2, 3\}$;
2. 3 flats of rank 1: $\{1\}$, $\{2\}$ and $\{3\}$;
3. 1 flat of rank 0: $\emptyset$.

Cohomology

Let $(C^*(Z/\sim;\mathbb{Z}), d)$ be the cellular complex of $Z/\sim$ defined by the CW complex structure explained above.
For a flat F of A, write ξ_F for the cellular cochain which is dual to equivalence class of parallel faces of Z that corresponds to F, so

$$C^*(Z/\sim;\mathbb{Z}) = \bigoplus_{F\text{: flat of } A} \mathbb{Z}\cdot\xi_F.$$

Cohomology

Theorem (Jiang-L.)

1. The cellular differential d is zero;
2. As graded $\mathbb{Z}$-modules, we have

$$H^*(Z/\sim;\mathbb{Z}) \cong \bigoplus_{F\text{: flat of } A} \mathbb{Z}\cdot[\xi_F],$$

   where $[\xi_F]$ is placed in degree $\mathrm{rk}(F)$;
3. For flats F, G of A, up to a sign, the cup product of $[\xi_F]$ and $[\xi_G]$ is given by

$$[\xi_F] \smile [\xi_G] = \begin{cases} [\xi_{F\vee G}] & \text{if } \mathrm{rk}(F)+\mathrm{rk}(G)=\mathrm{rk}(F\vee G) \\ 0 & \text{otherwise.} \end{cases}$$

Cohomology

Remarks

1. The last theorem is true for the complex locus of Y equipped with the Euclidean topology, except that $[\xi_F]$ is placed in degree $2\mathrm{rk}(F)$;
2. Let X be a projective variety defined over $\mathbb{R}$. It is very rare that $H^*(X(\mathbb{R});\mathbb{Z})$ is isomorphic to $H^*(X(\mathbb{C});\mathbb{Z})$ with degrees halved. In fact, the projective space $\mathbb{P}^m$, for $m \geq 2$, does not have this property.

$\mathbb{Z}/2$-equivariant cohomology ($\mathbb{Z}/2$ coefficients)

The group $\mathbb{Z}/2$ acts on $(\mathbb{P}^1(\mathbb{R}))^n$, where the nontrivial element acts by multiplying each component by -1.
It is evident that Y is stable under this action.

Theorem (Jiang-L.)

1. With $\mathbb{Z}/2$ coefficients, Y is $\mathbb{Z}/2$-equivariantly formal. In particular, we have an isomorphism

$$H^*_{\mathbb{Z}/2}(Y;\mathbb{Z}/2) \cong H^*(Y;\mathbb{Z}/2) \otimes_{\mathbb{Z}/2} (\mathbb{Z}/2)[s]$$

   of $(\mathbb{Z}/2)[s]$-modules;

2. For flats F, G of A, we have

$$\left([\xi_F] \otimes s^0\right) \smile \left([\xi_G] \otimes s^0\right) = [\xi_{F\vee G}] \otimes s^{\mathrm{rk}(F)+\mathrm{rk}(G)-\mathrm{rk}(F\vee G)}.$$

$\mathbb{Z}/2$-equivariant cohomology ($\mathbb{Z}$ coefficients)

Theorem (Jiang-L.)

1. $H^*_{\mathbb{Z}/2}(Y;\mathbb{Z})$ is concentrated in even degrees;
2. For each $k \in \mathbb{Z}_{\geq 0}$, we have
$$H^{2k}_{\mathbb{Z}/2}(Y;\mathbb{Z}) \cong H^{2k}(Y;\mathbb{Z}) \oplus \bigoplus_{i=1}^{2k} \frac{H^{2k-i}(Y;\mathbb{Z})}{2H^{2k-i}(Y;\mathbb{Z})} s^i;$$
3. For flats F, G of A and $a, b \in \mathbb{Z}_{\geq 0}$ with $\mathrm{rk}(F) + a, \mathrm{rk}(G) + b \in 2\mathbb{Z}$, we have
$$([\xi_F] \otimes s^a) \smile \left([\xi_G] \otimes s^b\right) = [\xi_{F\vee G}] \otimes s^{\mathrm{rk}(F)+\mathrm{rk}(G)-\mathrm{rk}(F\vee G)+a+b}.$$

Topological pseudomanifold

Theorem (Jiang-L.)

The space Y, with the stratification given by the skeleta of its CW complex structure, is a *topological pseudomanifold*. Moreover, the structure of the link at each point can be described combinatorially.

Example cont'd

For $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, the variety Y is smooth except at the point (∞, ∞, ∞).

From the picture below it is evident that the link at (∞, ∞, ∞) is $S^1 \sqcup S^1$.

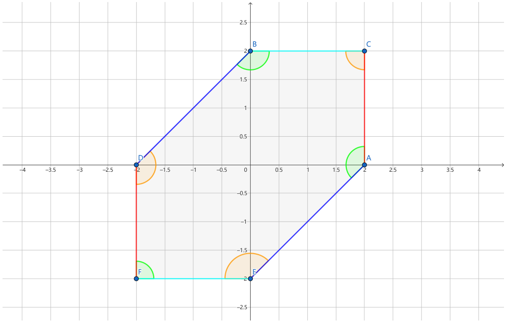

Figure: Neighborhood of (∞, ∞, ∞)

Coxeter arrangements

From now on we assume that A is a *Coxeter arrangement*, i.e. the rows of A are indexed by the roots of a root system Φ, the columns by a base $\{\alpha_1, \ldots, \alpha_d\}$ of Φ, and for $\alpha \in \Phi$ with

$$\alpha = c_1\alpha_1 + \cdots + c_d\alpha_d,$$

the row of A indexed by α is $[c_1 \ \ldots \ c_d]$.

In this case, the matroid Schubert variety Y is also called the *wonderful compactification of a Cartan subalgebra* [EL24].

Let W be the *Weyl group of* Φ. It is evident that the W-action on $(\mathbb{P}^1)^\Phi$ by permuting the components leaves Y stable. Hence, W acts on Y.

W-equivariant fundamental group

$G \curvearrowright X$: group action on a topological space, $x_0 \in X$: base point $\rightsquigarrow$

Definition

The G-equivariant fundamental group of (X, x_0) is

$$\pi_1^G(X, x_0) := \{(g, p) \colon p \text{ is a homotopy class of paths } x_0 \longrightarrow g \cdot x_0\}.$$

The multiplication in $\pi_1^G(X, x_0)$ is given by

$$(g_1, p_1) \cdot (g_2, p_2) = (g_1 g_2, p_1 * (g_1 \cdot p_2)).$$

W-equivariant fundamental group

$n \in \mathbb{Z}_{>0} \rightsquigarrow$
S_n: symmetric group (with generators s_i), Br_n: braid group (with generators σ_i) $\rightsquigarrow$

Definition

The *virtual braid group* VB_n is the free product $S_n * \mathrm{Br}_n$ modulo the relations

$$s_i s_{i+1} \sigma_i = \sigma_{i+1} s_i s_{i+1} \quad \forall i \in [1, n-1]$$
$$s_i \sigma_j = \sigma_j s_i \quad \forall i, j \in [1, n-1] \text{ with } |i-j| > 1.$$

The *virtual symmetric group* VS_n is $\mathrm{VB}_n / \langle \sigma_i^2 = 1 \colon i \in [1, n-1] \rangle$.
The *pure virtual symmetric group* PVS_n is

$$\ker \left(\mathrm{VS}_n \longrightarrow S_n, \ s_i, \sigma_i \longmapsto s_i \right).$$

W-equivariant fundamental group

Theorem (BEER [BEER06], IKLPR [IKL+24])

If A is of type A_n, then

$$\pi_1^{S_{n+1}}(Y) \cong \mathrm{VS}_n \quad \text{and} \quad \pi_1(Y) \cong \mathrm{PVS}_n.$$

Theorem (Jiang-L.)

Let A be a Coxeter arrangement, then

$$\pi_1^{W}(Y) \cong \mathrm{VW} \quad \text{and} \quad \pi_1(Y) \cong \mathrm{PVW}.$$

Remark

It is proved in [BEER06], [IKL+24] that, if A is of type A_n, the space Y is a CAT(0) space, hence a $K(\pi, 1)$ space. However, this is **NOT** true in general.

Positive geometry

Definition

The *totally nonnegative part* $Y_{\geq 0}$ *of* Y is

$$\{(x_i)_{i=1}^n \in Y \subseteq (\mathbb{P}^1(\mathbb{R}))^n \colon x_i \in \mathbb{R}_{\geq 0} \sqcup \{\infty\} \ \forall i \in [1, n]\}.$$

Theorem (IKLPR [IKL+24])

Let A be a Coxeter arrangement. The totally nonnegative part $Y_{\geq 0}$ is combinatorially isomorphic to a d-dimensional cube.

Positive geometry

Theorem (Jiang-L.)

Let A be a Coxeter arrangement. The triple

$$(Y, Y_{\geq 0}, \Omega := \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_d}{x_d})$$

is a *positive geometry* in the sense of Lam [Lam22].

Generalization to oriented matroids

Matrices with entries in $\mathbb{R}$ are precisely the *realizable oriented matroids*. For an arbitrary oriented matroid M, da Silva and Moulton [DM98] defined the *crinkled zonotope* Z_M *associated with* M.

We were able to generalize the equivalence relation $\sim$ on Z to this more general setting. Although the notion of matroid Schubert variety for a general oriented matroid is undefined, the quotient space $Z_M/\sim$ still makes sense.

Theorem (Jiang-L.)

All results above hold for $Z_M/\sim$.

In view of the last theorem, it is reasonable to call $Z_M/\sim$ the matroid Schubert variety associated to the (not-necessarily-realizable) oriented matroid M.

Example, cont'd

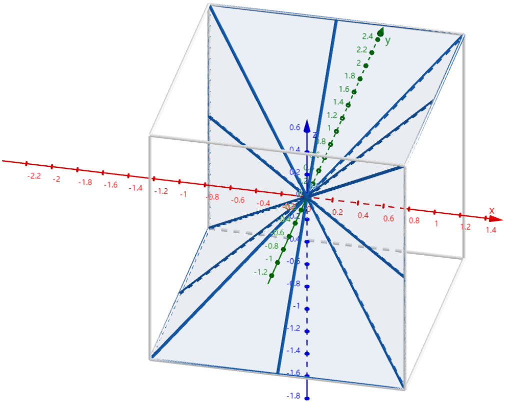

Figure: Crinkled zonotope for the oriented matroid of type A_2

Wilf's conjecture

$n, k \in \mathbb{Z}_{\geq 0} \rightsquigarrow S(n,k)$: *Stirling number of the second kind* (number of partitions of $[1, n]$ into k nonempty parts) $\rightsquigarrow B(n) := \sum_{k \geq 0} (-1)^k S(n,k)$: the *nth alternating Bell number* $\rightsquigarrow$

Wilf's conjecture

For any $n \in \mathbb{Z}_{\geq 0} \setminus \{2\}$, $B(n) \neq 0$.

Theorem (Jiang-L.)

Let A be the Coxeter arrangement of type A_n. We have

$$h^k(Y; \mathbb{Z}) = S(n+1, n-k+1) \quad \forall k \in \mathbb{Z}_{\geq 0}.$$

In particular, Wilf's conjecture holds if the Euler characteristic $\chi(Y)$ of Y is nonzero for all $n \in \mathbb{Z}_{\geq 0} \setminus \{1\}$.

A mysterious duality

The *Orlik-Solomon algebra associated with A* is the cohomology of

$$\mathbb{C}^d \setminus \bigcup_{i=1}^{n} (\ker(A_i) \otimes_{\mathbb{R}} \mathbb{C})\,.$$

For $k \in \mathbb{Z}_{\geq 0}$, let w_k (resp. W_k) be the *Whitney number of the first (resp. second) kind of A*.

The two kinds of Whitney numbers are combinatorially dual to each other.

Theorem

For any $k \in \mathbb{Z}_{\geq 0}$,

1. $h^k \left(\mathbb{C}^d \setminus \bigcup_{i=1}^{n} (\ker(A_i) \otimes_{\mathbb{R}} \mathbb{C})\,;\mathbb{Z} \right) = |w_k|$;
2. (Jiang-L.) $h^k(Y;\mathbb{Z}) = W_{n-k}$.

A mysterious duality

Question

Find geometric/topological relations between Y and $\mathbb{C}^d \setminus \bigcup_{i=1}^{n} (\ker(A_i) \otimes_{\mathbb{R}} \mathbb{C})$ that explains the combinatorial duality between the two kinds of Whitney numbers.

Koszulity of the cohomology of Y

Koszulity of the cohomology of Y

Characterize those matrices A such that $H^*(Y;\mathbb{Q})$ is a *Koszul algebra*.

With the exception of the root system of type F_4, we have proved the following result.

Theorem (Jiang-L.)

Let A be a Coxeter arrangement. The following are equivalent

1. $H^*(Y;\mathbb{Q})$ is Koszul;
2. $H^*(Y;\mathbb{Q})$ has a quadratic Gröbner basis;
3. A is *supersolvable*;
4. A is of types $A_n (n \geq 1)$, $B_n (n \geq 2)$ or G_2.

More open questions

Questions

1. Compute the intersection cohomology of Y;
2. Characterize those matrices A such that Y is a $K(\pi, 1)$ space;
3. $\cdots$.

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Thank you!