The difference calculus for functors on presheaves

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Calculus of differences

- **Aim:** Categorify Newton's difference operator Δ
  - For $f\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad \Delta[f](x) = f(x+1) - f(x)$
  - A discrete version of derivative

- **Inspired in part by:**
  - Work on polynomial functors by Kock [6], Niu/Spivak [7], and many others
  - Work on analytic functors by Joyal [5] *et. al.*
  - Multivariable analytic functors, e.g. Fiore/Gambino/Hyland/Winskel [4]
  - Differential structures, see Cockett/Cruttwell [3]

- **Likely related to:**
  - The cartesian difference categories of Alvarez-Picallo/Pacaud-Lemay [1]
  - The Goodwillie calculus, see e.g. Bauer/Johnson/Osborne/Riehl/Tebbe [2]

General idea

- For $F\colon \mathbf{Set} \longrightarrow \mathbf{Set}$, perturb the input and measure the difference in output

$$\Delta[F](X) = F(X+1) \setminus F(X)$$

Example

$F(X) = X^3$, then $F(X+1)$ has eight kinds of elements:

$$\begin{aligned}
&(x_1, x_2, x_3)\\
&(x_1, x_2, *), (x_1, *, x_3), (*, x_2, x_3)\\
&(x_1, *, *), (*, x_2, *), (*, *, x_3)\\
&(*, *, *)
\end{aligned}$$

$$\Delta[F](X) = 3X^2 + 3X + 1$$

Example

$F(X) = 2^X$ covariant power set, then $F(X+1)$ has two kinds of elements:

$$\begin{aligned}
&A \subseteq X \subseteq X+1\\
&A \cup \{*\} \subseteq X+1 \quad (A \subseteq X)
\end{aligned}$$

$$\Delta[F](X) = 2^X$$

Tautness

- $F(X+1)\setminus F(X)$ not always functorial

Definition

(Manes 2002) A functor is *taut* if it preserves inverse images

$$
\begin{array}{ccc}
A_0 & \rightarrowtail & A \\
{\scriptstyle f_0}\downarrow & \boxed{\text{Pb}} & \downarrow{\scriptstyle f} \\
B_0 & \rightarrowtail & B
\end{array}
\quad\Longrightarrow\quad
\begin{array}{ccc}
FA_0 & \rightarrowtail & FA \\
{\scriptstyle Ff_0}\downarrow & \boxed{\text{Pb}} & \downarrow{\scriptstyle Ff} \\
FB_0 & \rightarrowtail & FB
\end{array}
$$

A natural transformation $t\colon F \longrightarrow G$ is *taut* if the naturality squares corresponding to monos are pullbacks

$$
\begin{array}{ccc}
FA_0 & \rightarrowtail & FA \\
{\scriptstyle tA_0}\downarrow & \boxed{\text{Pb}} & \downarrow{\scriptstyle tA} \\
GA_0 & \rightarrowtail & GA
\end{array}
$$

- Get a sub-2-category $\mathscr{T}aut$ of $\mathscr{C}at$ whose objects are categories with inverse images and taut functors and taut natural transformations

Limits

Taut functors are closed under limits.

Proposition

(1) *Limits in* $\mathscr{C}at(\mathbf{Set}, \mathbf{Set})$ *of taut functors are taut.*

(2) *The inclusion*

$$\mathscr{T}aut(\mathbf{Set}, \mathbf{Set}) \rightarrowtail \mathscr{C}at(\mathbf{Set}, \mathbf{Set})$$

creates non-empty connected limits.

(3) *The product of taut functors is taut but the projections are not.*

Confluence

Theorem

$\mathbf{I}$ *colimits commute with inverse images in* $\mathbf{Set}$ *if and only if*

$$\forall \quad \begin{array}{ccc} & & I_1 \\ & \nearrow^{\alpha_1} & \\ I_0 & & \\ & \searrow_{\alpha_2} & \\ & & I_2 \end{array} \qquad \exists \quad \begin{array}{ccccc} & & I_1 & & \\ & \nearrow^{\alpha_1} & & \searrow^{\beta_1} & \\ I_0 & & & & I \\ & \searrow_{\alpha_2} & & \nearrow_{\beta_2} & \\ & & I_2 & & \end{array} \qquad \beta_1\alpha_1 = \beta_2\alpha_2\,.$$

Definition

If $\mathbf{I}$ satisfies the above conditions we say it's *confluent*.

Example

Filtered colimits, coproducts, quotients by group actions are all confluent.

Proposition

(1) *Confluent colimits in* $\mathscr{C}at(\mathbf{Set},\mathbf{Set})$ *of taut functors are taut.*

(2) $\mathscr{T}aut(\mathbf{Set},\mathbf{Set}) \rightarrowtail \mathscr{C}at(\mathbf{Set},\mathbf{Set})$ *creates all colimits.*

Examples

- Polynomial functors $P(X) = \sum_{i \in I} X^{A_i}$ are taut
- Analytic functors $\widetilde{F}(X) = \int^{n} X^n \times F(n) \cong \sum_n X^n \times F(n)/S_n$ are taut ($F\colon \mathbf{Bij} \longrightarrow \mathbf{Set}$ a species)
- Manes: Collection monads are finitary taut monads

The difference operator

Proposition

(1) *If* $F\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ *is taut then*

$$\Delta[F](X) = F(X+1) \setminus F(X)$$

defines a taut subfunctor of $F(X+1)$*.*

(2) *A taut transformation* $t\colon F \longrightarrow G$ *restricts to a taut transformation* $\Delta[t]\colon \Delta[F] \longrightarrow \Delta[G]$*.*

The functor

$$\Delta\colon \mathscr{T}\!aut(\mathbf{Set}, \mathbf{Set}) \longrightarrow \mathscr{T}\!aut(\mathbf{Set}, \mathbf{Set})$$

is called the *difference operator*.

Example

$\Delta[C] = 0$
$\Delta[X] = 1$

Colimits

Proposition

Δ *preserves colimits: For* $\Gamma : \mathbf{I} \longrightarrow \mathscr{T}aut(\mathbf{Set}, \mathbf{Set})$

$$\Delta[\varinjlim_{I} \Gamma I] \cong \varinjlim_{I} \Delta[\Gamma I]$$

Corollary

(1) $\Delta[F + G] \cong \Delta[F] + \Delta[G]$

(2) $\Delta[CF] \cong C\Delta[F]$

Limits

Proposition

$\Delta[F \times G] \cong (\Delta[F] \times G) + (F \times \Delta[G]) + (\Delta[F] \times \Delta[G])$.

More generally:

Proposition

$$\Delta\left[\prod_{i \in I} F_i\right] \cong \sum_{J \subsetneq I} \left(\prod_{j \in J} F_j\right) \times \left(\prod_{k \notin J} \Delta[F_k]\right) .$$

Theorem

Δ *preserves non-empty connected limits*

$$\Delta[\varprojlim_I \Gamma I] \cong \varprojlim_I \Delta[\Gamma I] .$$

Lax chain rule

Theorem

For taut functors F and G there is a taut natural transformation

$$\gamma_{G,F} \colon (\Delta[G] \circ F) \times \Delta[F] \longrightarrow \Delta[G \circ F]$$

which is:

(1) monic,

(2) natural in F and G,

(3) associative

$$\begin{array}{ccc}
(\Delta[H] \circ G \circ F) \times (\Delta[G] \circ F) \times \Delta[F] & \xrightarrow{\mathrm{id} \times \gamma_{G,F}} & (\Delta[H] \circ G \circ F) \times \Delta[G \circ F] \\
{\scriptstyle \gamma_{H,G \circ F} \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \gamma_{H,G \circ F}} \\
(\Delta[H \circ G] \circ F) \times \Delta[F] & \xrightarrow[\gamma_{H \circ G,F}]{} & \Delta[H \circ G \circ F]
\end{array} ,$$

(4) unitary

$$\begin{array}{ccc}
(\Delta[\mathrm{Id}] \circ F) \times \Delta[F] & \xrightarrow{\gamma_{\mathrm{Id},F}} & \Delta[\mathrm{Id} \circ F] \\
\| & & \| \\
1 \times \Delta[F] & \xrightarrow{\cong} & \Delta[F]
\end{array} , \qquad
\begin{array}{ccc}
(\Delta[F] \circ \mathrm{Id}) \times \Delta[\mathrm{Id}] & \xrightarrow{\gamma_{F,\mathrm{Id}}} & \Delta[F \circ \mathrm{Id}] \\
\| & & \| \\
\Delta[F] \times 1 & \xrightarrow{\cong} & \Delta[F]
\end{array} .$$

Tangent structure

For a taut functor F we define

$$\begin{array}{ccc} \mathbf{Set} \times \mathbf{Set} & \xrightarrow{T} & \mathbf{Set} \times \mathbf{Set} \\ {\scriptstyle P_1}\downarrow & & \downarrow{\scriptstyle P_1} \\ \mathbf{Set} & \xrightarrow[F]{} & \mathbf{Set} \end{array}$$

$$T(X,Y) = (FX, \Delta[F](X) \times Y)$$

Proposition

$T : \mathcal{Taut}(\mathbf{Set}, \mathbf{Set}) \longrightarrow \mathcal{Taut}(\mathbf{Set} \times \mathbf{Set}, \mathbf{Set} \times \mathbf{Set})$ *is a lax normal monoidal functor*

Polynomial functors

Proposition

If $P(X) = \sum_{i \in I} X^{A_i}$ *is a polynomial functor, then* $\Delta[P](X)$ *is again polynomial*

$$\Delta[P](X) \quad \cong \sum_{S \subsetneq A_i,\ i \in I} X^S$$

Example

$$\Delta[X^A] = \sum_{S \subsetneq A} X^S$$

Example

$$\Delta[X^n] = \sum_{k=0}^{n-1} \binom{n}{k} X^k$$

Multivariable functors

- Extend the difference calculus to functors

$$F\colon \mathbf{Set}^{\mathbf{A}} \longrightarrow \mathbf{Set}^{\mathbf{B}}$$

- $\mathbf{B}$ families of functors in $\mathbf{A}$ variables

- Partial difference with respect to A:

  For Φ in $\mathbf{Set}^{\mathbf{A}}$, perturb it by adding a single element of type A freely,
  $\Phi \rightsquigarrow \Phi + \mathbf{A}(A,-)$

$$\Delta_A[F](\Phi) = F(\Phi + \mathbf{A}(A,-)) \setminus F(\Phi)$$

- The one-variable theory carries over with some modifications

- Based on profunctors

Profunctors (a.k.a. 2-matrices)

- A profunctor $P\colon \mathbf{A} \longrightarrow\!\!\!\!\!\bullet\,\,\mathbf{B}$ is a functor $P\colon \mathbf{A}^{op} \times \mathbf{B} \longrightarrow \mathbf{Set}$
  A morphism of profunctors is a natural transformation

- P can be thought of as a $\mathbf{B}$ by $\mathbf{A}$ matrix of sets

- Composition of $P\colon \mathbf{A} \longrightarrow\!\!\!\!\!\bullet\,\,\mathbf{B}$ with $Q\colon \mathbf{B} \longrightarrow\!\!\!\!\!\bullet\,\,\mathbf{C}$ is "matrix multiplication"

$$(Q \otimes P)(A, C) = \int^{B} Q(B, C) \times P(A, B)$$

- Identities are hom functors

$$\mathrm{Id}_{\mathbf{A}} = \mathbf{A}(-,-)\colon \mathbf{A}^{op} \times \mathbf{A} \longrightarrow \mathbf{Set}$$

2-vectors (a.k.a. presheaves)

- A profunctor $\mathbb{1} \longrightarrow\!\!\!\!\!\!\bullet\; \mathbf{A}$ is a functor $\mathbb{1}^{op} \times \mathbf{A} \longrightarrow \mathbf{Set}$ which we identify with the presheaf $\Phi \in \mathbf{Set}^{\mathbf{A}}$

- Composing Φ with a profunctor P gives an object $P \otimes \Phi$ of $\mathbf{Set}^{\mathbf{B}}$ and so we get a functor $P \otimes (\ ) \colon \mathbf{Set}^{\mathbf{A}} \longrightarrow \mathbf{Set}^{\mathbf{B}}$ which is cocontinuous (2-linear)

- Its partial difference with respect to A is

$$
\begin{aligned}
\Delta_A[P \otimes (\ )](\Phi) &= P \otimes (\Phi + \mathbf{A}(A,-)) \setminus P \otimes \Phi \\
&\cong (P \otimes \Phi + P \otimes \mathbf{A}(A,-)) \setminus P \otimes \Phi \\
&\cong P \otimes \mathbf{A}(A,-) \\
&\cong P(A,-)
\end{aligned}
$$

  a constant functor (independent of Φ)

$$\mathbf{Set}^{\mathbf{A}} \longrightarrow \mathbf{Set}^{\mathbf{B}}$$

Tense functors

$P \otimes (\ )$ is not taut!

Definition

F is *tense* if it preserves complemented subobjects and their pullbacks

$t\colon F \longrightarrow G$ is *tense* if the naturality squares corresponding to complemented subobjects are pullbacks

- If F preserves binary coproducts then it's tense, so $P \otimes (\ )$ is tense
- There is a sub-2-category of $\mathscr{C}at$, $\mathscr{T}ense$, consisting of presheaf categories, tense functors and tense natural transformations

Limits and colimits

Proposition

(1) *Let* $\Gamma\colon \mathbf{I} \longrightarrow \mathscr{C}at(\mathbf{Set}^{\mathbf{A}}, \mathbf{Set}^{\mathbf{B}})$ *be such that* $\Gamma(I)$ *is tense for every* $\mathbf{I}$*. Then* $\varprojlim \Gamma$ *is also tense. If* $\mathbf{I}$ *is confluent so is* $\varinjlim \Gamma$*.*

(2) $\mathscr{T}ense(\mathbf{Set}^{\mathbf{A}}, \mathbf{Set}^{\mathbf{B}}) \rightarrowtail \mathscr{C}at(\mathbf{Set}^{\mathbf{A}}, \mathbf{Set}^{\mathbf{B}})$ *creates non-empty connected* $\varprojlim$ *and all* $\varinjlim$*.*

Partial difference

Proposition

Let $F\colon \mathbf{Set}^{\mathbf{A}} \longrightarrow \mathbf{Set}^{\mathbf{B}}$ *be tense, then*

$$\Delta_A[F](\Phi) = F(\Phi + \mathbf{A}(A,-)) \setminus F(\Phi)$$

defines a tense subfunctor

$$\Delta_A[F] \rightarrowtail F(- + \mathbf{A}(A,-))$$

functorial in F

$$\Delta_A\colon \mathscr{T}ense\,(\mathbf{Set}^{\mathbf{A}},\mathbf{Set}^{\mathbf{B}}) \longrightarrow \mathscr{T}ense\,(\mathbf{Set}^{\mathbf{A}},\mathbf{Set}^{\mathbf{B}})\;.$$

Definition

$\Delta_A[F]$ is the *partial difference* of F with respect to A.

- $\Delta_A[C] = 0$
- $\Delta_A[P \otimes (\;)] \cong P(A,-)$ (constant)

Limits and colimits

Proposition

$\Delta_A \colon \mathscr{T}ense\ (\mathbf{Set}^{\mathbf{A}}, \mathbf{Set}^{\mathbf{B}}) \longrightarrow \mathscr{T}ense\ (\mathbf{Set}^{\mathbf{A}}, \mathbf{Set}^{\mathbf{B}})$ *preserves colimits and non-empty connected limits*

Corollary

(1) $\Delta_A[F + G] \cong \Delta_A[F] + \Delta_A[G]$
(2) $\Delta_A[C \times F] \cong C \times \Delta_A[F]$

Proposition

$$\Delta_A \left[\prod_{i \in I} F_i \right] \cong \sum_{J \subsetneq I} \left(\prod_{j \in J} F_j \right) \times \left(\prod_{k \notin J} \Delta_A[F_k] \right)$$

Corollary

$$\Delta_A[F \times G] \cong (\Delta_A[F] \times G) + (F \times \Delta_A[G]) + (\Delta_A[F] \times \Delta_A[G])$$

(Discrete) Jacobian

For $F\colon \mathbf{Set}^{\mathbf{A}} \longrightarrow \mathbf{Set}^{\mathbf{B}}$ a tense functor

Proposition

For Φ in $\mathbf{Set}^{\mathbf{A}}$, $\Delta_A[F](\Phi)$ is (contravariantly) functorial in A

$$\Delta[F](\Phi)\colon \mathbf{A}^{op} \longrightarrow \mathbf{Set}^{\mathbf{B}}$$

- $\Delta[F](\Phi)$ is a profunctor $\mathbf{A} \longrightarrow\!\!\!\!\!\bullet\;\, \mathbf{B}$, the *(discrete) Jacobian* of F at Φ

Proposition

$\Delta[F](\Phi)$ is functorial in Φ giving a tense functor

$$\Delta[F]\colon \mathbf{Set}^{\mathbf{A}} \longrightarrow \mathbf{Set}^{\mathbf{A}^{op}\times\mathbf{B}} = \mathscr{P}rof(\mathbf{A},\mathbf{B})$$

Proposition

$\Delta[F]$ is functorial in F giving the Jacobian functor

$$\Delta\colon \mathscr{T}ense\,(\mathbf{Set}^{\mathbf{A}},\mathbf{Set}^{\mathbf{B}}) \longrightarrow \mathscr{T}ense\,(\mathbf{Set}^{\mathbf{A}},\mathbf{Set}^{\mathbf{A}^{op}\times\mathbf{B}})$$

Alternate formulations

- **Differential operator**

$$D[F] \colon \mathbf{Set}^{\mathbf{A}} \times \mathbf{Set}^{\mathbf{A}} \longrightarrow \mathbf{Set}^{\mathbf{B}}$$

$$D[F](\Phi, \Psi) = \Delta[F](\Phi) \otimes \Psi$$

$D[F]$ is cocontinuous in the second variable

- **Tangent functor**

$$\begin{array}{ccc} \mathbf{Set}^{\mathbf{A}} \times \mathbf{Set}^{\mathbf{A}} & \xrightarrow{T[F]} & \mathbf{Set}^{\mathbf{B}} \times \mathbf{Set}^{\mathbf{B}} \\ {\scriptstyle P_1}\downarrow & & \downarrow{\scriptstyle P_1} \\ \mathbf{Set}^{\mathbf{A}} & \xrightarrow[F]{} & \mathbf{Set}^{\mathbf{B}} \end{array}$$

$$T[F](\Phi, \Psi) = (F(\Phi), \Delta[F](\Phi) \otimes \Psi)$$

$T[F]$ also cocontinuous in the second variable

Lax chain rule

Theorem

For tense functors $F\colon \mathbf{Set}^{\mathbf{A}} \longrightarrow \mathbf{Set}^{\mathbf{B}}$, $G\colon \mathbf{Set}^{\mathbf{B}} \longrightarrow \mathbf{Set}^{\mathbf{C}}$ *and* Φ *in* $\mathbf{Set}^{\mathbf{A}}$ *we have a canonical comparison*

$$\gamma\colon \Delta[G](F(\Phi)) \otimes_{\mathbf{B}} \Delta[F](\Phi) \longrightarrow \Delta[GF](\Phi)$$

which is
(1) natural in Φ
(2) natural in F *and* G
(3) associative
(4) normal

Corollary

$T\colon \mathscr{T}ense \longrightarrow \mathscr{T}ense$

$$\begin{array}{ccc} \mathbf{Set}^{\mathbf{A}} & & \mathbf{Set}^{\mathbf{A}} \times \mathbf{Set}^{\mathbf{A}} \\ {\scriptstyle F}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle T[F]} \\ \mathbf{Set}^{\mathbf{B}} & & \mathbf{Set}^{\mathbf{B}} \times \mathbf{Set}^{\mathbf{B}} \end{array}$$

is a lax normal functor

Multivariable analytic functors

After Fiore *et al.* [4]

- $!\mathbf{A}$ free symmetric monoidal category generated by $\mathbf{A}$
  - Objects: finite sequences $\langle A_1, \ldots, A_n \rangle$
  - Morphisms: $(\sigma, \langle f_1, \ldots, f_m \rangle) \colon \langle A_1, \ldots, A_n \rangle \longrightarrow \langle A'_1, \ldots, A'_m \rangle$
    $\sigma \colon m \longrightarrow n$ bijection, $f_i \colon A_{\sigma i} \longrightarrow A'_i$
- $\mathbf{A}$-$\mathbf{B}$ *symmetric sequence* (multivariable species) is a profunctor $P \colon !\mathbf{A} \longrightarrow\!\!\!\bullet\; \mathbf{B}$
- Defines a *multivariable analytic functor*

$$\widetilde{P} \colon \mathbf{Set}^{\mathbf{A}} \longrightarrow \mathbf{Set}^{\mathbf{B}}$$

$$\widetilde{P}(\Phi)(B) = \int^{\langle A_1 \ldots A_n \rangle \in !\mathbf{A}} P(A_1, \ldots, A_n; B) \times \Phi A_1 \times \cdots \times \Phi A_n$$

Theorem
$\widetilde{P}$ is tense and $\Delta[\widetilde{P}]$ is an analytic functor $\mathbf{Set}^{\mathbf{A}} \longrightarrow \mathbf{Set}^{\mathbf{A}^{op} \times \mathbf{B}}$

The difference symmetric sequence

$\Delta[\widetilde{P}] \cong \widetilde{Q}$ for $Q\colon !\mathbf{A} \longrightarrow\!\!\!\!\!\bullet\ \mathbf{A}^{op} \times \mathbf{B}$

$$Q(A_1,\ldots,A_n;A,B) = \sum_{k=1}^{\infty} P(A_1,\ldots,A_n,A,\ldots,A;B)/\{\mathrm{id}_n\} \times S_k$$

where there are k A's in the k^{th} summand

When $\mathbf{A} = \mathbf{B} = \mathbb{1}$, $!\mathbf{A} \cong \mathbf{Bij}$ and we recover the original definition of species and analytic functor. Then

$$Q\colon \mathbf{Bij} \longrightarrow \mathbf{Set}$$

$$Q(n) = \sum_{k=1}^{\infty} P(n+k)/\{\mathrm{id}\} \times S_k$$

A Q-structure on n is a positive integer k and an equivalence class of P-structures on $n+k$, two structures being equivalent if there is a permutation of $n+k$ fixing the first n elements which transforms one into the other

Exponential functors

- How should we categorify $f(x) = a^x$, $a > 0$?

Example

$F(X) = 2^X$ covariant power set

If L is a sup-lattice we can make $F(X) = L^X$ into a covariant functor $L^X : \mathbf{Set} \longrightarrow \mathbf{Set}$ by Kan extension. For $f : X \longrightarrow Y$ and $\phi \in L^X$

$$F(f)(\phi)(y) = \bigvee_{f(x)=y} \phi(x) \, .$$

Proposition

$L^X : \mathbf{Set} \longrightarrow \mathbf{Set}$ *is taut and*

$$\Delta[L^X] \cong L_* \times L^X$$

where $L_* = L \setminus \{\bot\}$ *.*

Example

$\Delta[3^X] \cong 2 \times 3^X$

Dirichlet functors?

- A first try might be

$$F(X) = \sum_{i \in I} L_i^X$$

- For every positive integer n the ordinal

$$\mathbf{n} = \{1 < 2 < 3 < \cdots < n\}$$

is a sup-lattice, but ...

- For any unbounded sequence $n_1 < n_2 < \ldots$

$$\sum_{i \in \mathbb{N}} \mathbf{n}_i^X \cong \sum_{n \in \mathbb{N}} \mathbf{n}^X$$

Normalized exponentials

- L^X is not connected: $\pi_0(L^X) \cong L$

$$L^X \cong \sum_{l \in L} C_l(X) \quad C_l(X) = \{f\colon X \longrightarrow L \mid \bigvee f(x) = l\}$$

- The *normalized exponential*

$$L^{[X]} = \{f\colon X \longrightarrow L \mid \bigvee f(x) = \top\}$$

- $L^X = \sum_{l \in L} (L/l)^{[X]} \quad L/l = \{l' \in L \mid l' \leq l\}$

Proposition

$L^{[X]}$ is taut and

$$\Delta\left[L^{[X]}\right] \cong \sum_{\substack{l \vee l' = \top \\ l' \neq \bot}} (L/l)^{[X]}$$

Corollary

If $\top$ is join irreducible ($i.e.\ l \vee l' = \top \Rightarrow l = \top$ or $l' = \top$) then

$$\Delta\left[L^{[X]}\right] \cong L_* \times L^{[X]} + \sum_{l \neq \top} (L/l)^{[X]}$$

(Covariant) Dirichlet functors

Proposition

If $\langle L_i \rangle_{i \in I}$ *and* $\langle M_j \rangle_{j \in J}$ *are two families of sup-lattices such that*

$$\sum_{i \in I} L_i^{[X]} \cong \sum_{j \in J} M_j^{[X]}$$

then there is a bijection $\alpha \colon I \longrightarrow J$ *and lattice isomorphisms*

$$L_i \cong M_{\alpha(i)} \, .$$

Definition

A *(covariant) Dirichlet functor* is a functor of the form

$$F(X) = \sum_{i \in I} L_i^{[X]}$$

for $\langle L_i \rangle$ a family of sup-lattices.

Dirichlet difference

Proposition

Dirichlet functors are taut and closed under products and coproducts

Theorem

If $F(X) = \sum_{i \in I} L_i^{[X]}$ *is Dirichlet, then so is* $\Delta[F](X)$ *and*

$$\Delta[F](X) = \sum_{i \in I, l \in L_i} C_l \times (L_i / l)^{[X]}$$

where $C_l = \{l' \in L_i \mid l' \neq \bot \ \wedge \ l \vee l' = \top\}$

References

[1] Mario Alvarez-Picallo and Jean-Simon Pacaud-Lemay.
Cartesian difference categories.
Logical Methods in Computer Science, 17(3):23:1–23:48, 2021.

[2] Kristine Bauer, Brenda Johnson, Christina Osborne, Emily Riehl, and Amelia Tebbe.
Directional derivatives and higher order chain rules for abelian functor calculus.
Topology and its Applications, 235:375–427, 2018.

[3] R. Cockett and G. Cruttwell.
Differential structure, tangent structure, and SDG.
Applied Categorical Structures, 22(2):331–417, 2014.

[4] M. Fiore, N. Gambino, M. Hyland, and G. Winskel.
The cartesian closed bicategory of generalised species of structures.
Journal of the London Mathematical Society, 77(1):203–220, February 2008.

[5] André Joyal.
Une théorie combinatoire des séries formelles.
Advances in Mathematics, 42(1):1–82, October 1981.

[6] Joachim Kock.
Notes on polynomial functors.
`http://mat.uab.cat/~kock/cat/polynomial.html`, August 16 2016.

[7] Nelson Niu and David I. Spivak.
Polynomial functors: A mathematical theory of interaction.
`https://github.com/ToposInstitute/poly`, October 11 2023.