Undecidability and Free Adjoints

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Abstract

We exhibit a free construction for adding adjoints to categories. The word problem for 2-cells is in general undecidable; in particular, it is undecidable for **Grp**, **Set** and many others. However for some categories we show that it is decidable.

Keywords 2-category, adjoints, undecidability.

1 Introduction

A *2-category* is a category whose arrows are themselves the objects of another category, the arrows of which (known as *2-cells*) obey certain naturality conditions. A standard example is the 2-category **Cat** of categories, functors, and natural transformations. As Eilenberg and Mac Lane [4, p.18] made clear:

> "[C]ategory" has been defined in order to be able to define "functor" and "functor" has been defined in order to be able to define "natural transformation".

the need for this structure on **Cat** was understood even before the explicit introduction of 2-categories.

In particular, the 2-cell structure of **Cat** permitted the introduction of the concept of *adjoint functors*. Examples of adjoint functors in **Cat** include free functors, the hom-tensor adjunction, and various adjunctions between logical operations. Adjunction is also important in the definition of *Cartesian closed categories* which are related to the lambda calculus. An arrow $f\colon a \to b$ is said to be *left adjoint* to $g\colon b \to a$ (and g *right adjoint* to f) if there exist 2-cells $\eta\colon I_b \Rightarrow fg$ and $\varepsilon\colon gf \Rightarrow I_a$ such that $(\varepsilon g)(g\eta) = i_g$ and $(f\varepsilon)(\eta f) = i_f$. It was only much later that it was realized that adjunctions are important in other 2-categories as well.

Various familiar constructions can be interpreted as adding right (or left) adjoints to every arrow in a given category. For example, the category of relations and the category of spans can be obtained from the category of sets by adding right adjoints subject to certain conditions. We have described a construction Π_2 which adds right adjoints to all arrows in a category in a free way, which means that the resulting category will satisfy the conditions following from the fact that the new arrows are adjoints to the arrows in the old category, and no other conditions. (See [6] for a special case.) This construction is free, in the sense that every other construction that adds adjoints factors uniquely (up to 2-isomorphism) through this one.

2 Freely Adding Adjoints

In [2] we have studied the structure of the 2-category $\Pi_2\mathcal{C}$ obtained by freely adding adjoints to all arrows in a 2-category $\mathcal{C}$. In this paper we are interested in the results of this construction for the case where $\mathcal{C}$ is a category. We start by summarizing the results of [2] for this special case.

2.1 The 2-category $\Pi_2\mathcal{C}$.

Let $\mathcal{C}$ be any category. We write $\Pi_2\mathcal{C}$ for the following 2-category:

The **objects** of $\Pi_2\mathcal{C}$ are the same as the objects of $\mathcal{C}$, so $\Pi_2\mathcal{C}_0 = \mathcal{C}_0$.

The **arrows** in $\Pi_2\mathcal{C}$ are zig-zags in $\mathcal{C}$, i.e. an arrow in $\mathcal{C}(A,B)$ is of the form

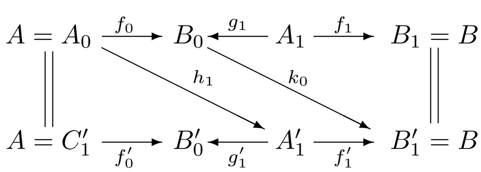
$$A = A_0 \stackrel{f_0}{\to} B_0 \stackrel{g_1}{\leftarrow} A_1 \stackrel{f_1}{\to} B_1 \cdots \stackrel{g_n}{\leftarrow} A_n \stackrel{f_n}{\to} B_n = B.$$

We denote such an arrow by $(g_1, \cdots, g_n; f_0, \cdots, f_n)$, except when $n = 0$, in which case we will write (f_0).

A **2-cell**

$$(g_1, \cdots, g_n; f_0, \cdots, f_n) \Rightarrow (g'_1, \cdots, g'_m; f'_0, \cdots, f'_m)$$

is an equivalence class of diagrams. Examples of such diagrams are



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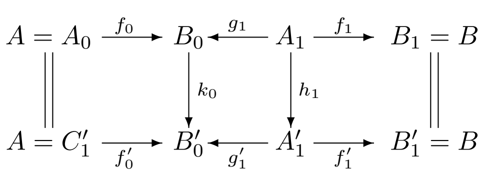


Specifically, these diagrams are determined by an adjoint pair of order-preserving index functions $\phi: \{0, \cdots, m\} \to \{0, \cdots, n\}$ and $\psi: \{0, \cdots, n\} \to \{0, \cdots, m\}$ such that $\psi\phi(j) \geq j$ and $\phi\psi(i) \leq i$, together with families of arrows

$$k_i: B_i \to B'_{\psi(i)} \text{ and } h_j: A_{\phi(j)} \to A'_j,$$

such that all resulting squares commute, i.e.

$$f'_j \circ h_j = \begin{cases} g'_{j+1} \circ h_{j+1} & \text{if } \phi(j) = \phi(j+1) \\ k_{\phi(j)} \circ f_{\phi(j)} & \text{if } \phi(j) \neq \phi(j+1) \end{cases}$$

for $j = 1, \ldots, m$;

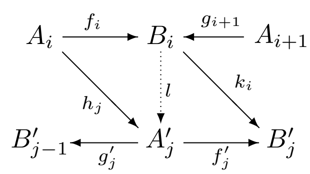
$$k_i \circ g_{i+1} = \begin{cases} k_{i+1} \circ f_{i+1} & \text{if } \psi(i) = \psi(i+1) \\ g'_{\psi(i)+1} \circ h_{\psi(i)+1} & \text{if } \psi(i) \neq \psi(i+1) \end{cases}$$

for $i = 1, \ldots, n$.

Finally we require that $h_0 = \mathrm{id}_A$ and $k_n = \mathrm{id}_B$. We denote such a representative for a 2-cell by $(\phi; \psi; k_0, \ldots, k_{n-1}; h_1, \ldots, h_m)$ or $(\phi; \psi; (k_i); (h_j))$. Note that in the example above, $\psi(0) = 1$, $\psi(1) = 1$ and $\psi(2) = 3$, and $\phi(0) = 0$, $\phi(1) = 0$, $\phi(2) = 2$, and $\phi(3) = 2$.

2.2 The equivalence relation

The *equivalence relation* on these diagrams is generated (through symmetry and transitivity) by the following relationship: Two diagrams θ and ω representing 2-cells in $\Pi_2\mathcal{C}$ are called *directly related* and we write $\theta \sim_1 \omega$ if they are everywhere the same except for two parts with the following shapes: the cells ω and θ contain



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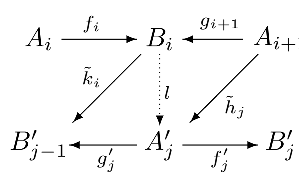

respectively; and there exists an arrow $l: B_i \to A'_j$ which factors both diagrams. Such an arrow l will be said to *link* θ and ω. We will postpone further discussion of this equivalence relation to the next section.

2.3 Composition

Composition of arrows in $\Pi_2\mathcal{C}$ is defined by concatenating of the zig-zags and composing the two composable morphisms. The definition of horizontal and vertical composition of 2-cells can be found in [2], but we won't explicitly use it in this article.

2.4 Universal Property of $\Pi_2\mathcal{C}$

There is an embedding functor $(-)_*: \mathcal{C} \to \Pi_2\mathcal{C}$, which is the identity on objects and sends morphisms to sequences of length 1: $(f)_* = (f)$. In order to describe the properties of this functor, we need the following definition:

Definition 2.1 *Let $\mathcal{D}$ be a 2-category. A functor $F: \mathcal{C} \to \mathcal{D}$ is called* sinister *if it sends arrows in $\mathcal{C}$ to left adjoints (i.e. arrows with a right adjoint) in $\mathcal{D}$.*

Recall (cf. [1]) that a natural transformation α between two 2-functors $F, G: \mathcal{C} \rightrightarrows \mathcal{D}$ can be represented as a 2-functor $K_\alpha: \mathcal{C} \to \mathrm{Cyl}(\mathcal{D})$. So we call a natural transformation α *sinister* if the 2-functor K_α is sinister. Note that if α is sinister, then its domain and codomain are sinister too, but the converse is not necessarily true. We will write $\mathrm{Sin}(\mathcal{C}, \mathcal{D})$ for the 2-category of sinister 2-functors, sinister transformations, and sinister modifications.

With this notation, the functor $(-)_*$ has the following universal property (cf. [2]).

Theorem 2.2 *The functor $(-)_*: \mathcal{C} \to \Pi_2\mathcal{C}$ has the property that for each arrow $f \in \mathcal{C}_1$ the arrow $f_* \in \Pi_2\mathcal{C}_1$ has a right adjoint in $\Pi_2\mathcal{C}$ (i.e., f_* is a left adjoint). Moreover, this functor is universal with this property in the sense that composition with $(-)_*$ defines an equivalence of categories*

$$\mathrm{Hom}(\Pi_2\mathcal{C}, \mathcal{D}) \to \mathrm{Sin}(\mathcal{C}, \mathcal{D}),$$

3 Decidability and Undecidability

The purpose of this section is twofold: we describe a class of commonly used categories for which the equivalence relation described before would be undecidable, and we discuss some conditions under which the equivalence relation becomes decidable, and give examples of categories that satisfy these conditions.

3.1 Undecidability Results

It must be remembered that decidability problems in general concern representations of mathematical structures, not the structures themselves. Thus for instance, the word problem for a group is trivial given a complete multiplication table for the group. In the undecidability results that follow we will assume a graph to be equipped with a bounded time algorithm to determine the existence of an edge joining any two specified vertices.

Given an arbitrary 2-register abacus (see [3]) with internal states S_i, we will construct a bipartite graph as follows. Let U be the set of triples $u_i = (S_i, X_i, Y_i)$ and V is the set of legal transitions $v_{ij}: (S_i, X_i, Y_i) \to (S_j, X_j, Y_j)$. The vertex v_{ij} is joined to both u_i and u_j and there are no other edges. The undecidability of the halting problem for the 2-register abacus [5], implies the following result (which is probably a folk theorem):

Lemma 3.1 *The problem of deciding whether two vertices in an infinite bipartite graph are connected by a finite path is undecidable.*

Theorem 3.2 *The equivalence relation on the 2-cells of the category $\Pi_2\mathbf{Set}$ is undecidable.*

Proof Let $V = V_1 \cup V_2$ and E be the sets of vertices and edges respectively for a bipartite graph (i.e., $V_1 \cap V_2 = \emptyset$ and every edge has an endpoint in V_1 and an endpoint in V_2) for which in general it is undecidable whether two of its vertices are connected by a path of finite length. For $i = 1, 2$, let $p_i: E \to V_i$ be the function that sends an edge to its endpoint in V_i.

Representatives for 2-cells in $\Pi_2\mathbf{Set}$ of the form

$$\begin{array}{ccccccc} \emptyset & \longrightarrow & \{*\} & \longleftarrow & \emptyset & \longrightarrow & V_2 \\ \| & & \downarrow v & & \downarrow & & \| \\ \emptyset & \longrightarrow & V_1 & \xleftarrow[p_1]{} & E & \xrightarrow[p_2]{} & V_2 \end{array}$$

are in one-to-one correspondence with vertices in V_1. It is not difficult to see that two of these are equivalent if and only if there is a path of finite length connecting them in the graph.

Corollary 3.3 *If a category $\mathcal{C}$ contains the category* **Set** *as a full subcategory, the equivalence relation to define $\Pi_2\mathcal{C}_2$ is undecidable.*

Examples A simple modification of this applies to $\mathbf{Set}_*$: the diagram in the proof of the theorem can easily be adjusted to contain only nonempty spaces and pointed maps. It also applies to topological categories such as the category **Top** of topological spaces, the category $\mathbf{Top}_*$ of pointed topological spaces, and the category **Man** of manifolds.

Theorem 3.4 *The equivalence relation on the 2-cells of the category $\Pi_2\mathbf{Grp}$ is undecidable.*

Proof We prove this by showing that one can simulate the word problem for groups in this equivalence relation, i.e., for every group defined by generators and relations, there are arrows in $\Pi_2\mathbf{Grp}$ such that representatives for 2-cells between them correspond to words in the group and they represent equivalent 2-cells precisely when the words represent the same group element.

Let $G(X, R)$ be the group generated by a set X with a set of relations $R = \{w_i = w_i' | i \in I\}$ indexed by a set I, where the w_i and w_i' are words in the free group FX. Let $p_1 : F(X \dot\cup I) \to FX$ be the group homomorphism that sends $i \in I$ to w_i and is the identity on X. Analogously, define $p_2 : F(X \dot\cup I) \to FX$ to be the group homomorphism that sends $i \in I$ to w_i' and is the identity on X. Consider the diagram, representing a 2-cell in $\Pi_2(\mathbf{Grp})$:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z} & \longleftarrow & 0 & \longrightarrow & FX \\
\| & & \downarrow{\scriptstyle v} & & \downarrow & & \| \\
0 & \longrightarrow & FX & \xleftarrow[p_1]{} & F(X \dot\cup R) & \xrightarrow[p_2]{} & FX
\end{array}
$$

The arrow v is uniquely determined by the word $v(1) \in FX$, and two such 2-cell representatives are equivalent precisely when the corresponding words are equal in G.

3.2 Decidability Conditions

If the category $\mathcal{C}$ is locally finite, equality of any two given 2-cells in $\Pi_2\mathcal{C}$ can be settled by exhaustive checking of finitely many hom-sets. Thus, the problem is decidable for categories such as finite sets, finite groups, or any poset. Again, if the arrows of $\mathcal{C}$ satisfy appropriate cancellation conditions, then no square can factorize in more than one way; for example, the problem is decidable for any category whose arrows are all epic or all monic.

There are however more subtle conditions that also guarantee decidability. Suppose that if any two 2-cell representatives are equivalent by a long sequence of direct equivalences, there is always a way to shorten the sequence to one of a length bounded by an expression in the lengths of the domain and and the codomain of the 2-cell. For such a category $\mathcal{C}$ only a finite number of sequences of equivalences would need to be examined to determine whether two 2-cells were equal, leading again to decidability.

It is conjectured that the categories $\mathbf{Ab}_{fg}$ of finitely generated abelian groups and $\mathbf{Vect}$ of vector spaces satisfy conditions of this type. As an example of the techniques that might be used to prove this, we show a partial result: that the equality of short enough 2-cells in $\Pi_2(\mathbf{Ab}_{fg})$ is decidable (an identical proof works for $\Pi_2(\mathbf{Vect})$).

Proposition 3.5 *Let* $A_0 \overset{f_0}{\to} B_0 \overset{g_1}{\leftarrow} A_1 \overset{f_1}{\to} B_1$ *and* $A_0 \overset{f_0'}{\to} B_0' \overset{g_1'}{\leftarrow} A_1' \overset{f_1'}{\to} B_1$ *be two arrows of* $\Pi_2(\mathbf{Ab}_{fg})$. *The equality of two representatives for 2-cells* $(g_1; f_0, f_1) \Rightarrow (g_1'; f_0', f_1')$ *of* $\Pi_2(\mathbf{Ab}_{fg})$ *is decidable.*

Proof Consider the typical case (the others can be treated similarly) in which both index functions of each representative are identities. In this case we may write $\alpha = (k_0, h_1)$ and $\alpha' = (k_0', h_1')$, with $k_0, k_0' : B_0 \to B_0'$ and $h_1, h_1' : A_1 \to A_1'$. Suppose that there exists a sequence of $2n$ $(n > 2)$ direct equivalences

$$\alpha = \alpha_1 \sim_1 \beta_1 \sim_1 \alpha_2 \sim_1 \cdots \sim_1 \alpha_n = \alpha',$$

where α_i is linked to β_i by γ_i and β_i is linked to α_{i+1} by δ_i. Therefore,

$$
\begin{aligned}
\gamma_1 g_1 &= h_1, & g_1'\gamma_1 &= k_0, \\
f_1'\delta_i &= f_1'\gamma_i, & \delta_i f_0 &= \gamma_i f_0 \\
& & &\text{for } i = 1, \ldots, n; \\
g_1'\delta_i &= g_1'\gamma_{i+1}, & \delta_i g_1 &= \gamma_{i+1} g_1 \\
& & &\text{for } i = 1, \ldots, n-1; \\
g_1'\delta_n &= k_0', & \text{and } \delta_n g_1 &= h_1'.
\end{aligned}
$$

Suppose that $n > 2$, and let $\Gamma : B_0 \to A_1'$ be the homomorphism that takes $b \in B_0$ to $\gamma_1(b) - \delta_1(b) + \gamma_2(b) - \cdots + \gamma_n(b)$. Thus for $a \in A_1$, $\Gamma \circ g_1(a) = \gamma_1(g_1(a)) - \delta_1(g_1(a)) + \gamma_2(g_1(a)) - \cdots - \delta_{n-1}(g_1(a)) + \gamma_n(g_1(b))$. As $\delta_i(g_1(a)) = \gamma_{i+1}(g_1(a))$, this sum equals $\gamma_1(g_1(a))$ which equals $h_1(a)$; so $\Gamma \circ g_1 = h_1$. Similar arguments show that $g_1' \circ \Gamma = k_0$, $\Gamma \circ f_0 = h_{n-1}$, and $f_1' \circ \Gamma = k_{n-1}$, where $h_{n-1} : A_0 \to A_1'$ and $k_{n-1} : B_0 \to B_1$ are the components for β_{n-1}. Therefore α is linked directly to β_{n-1} by Γ, reducing the chain to two links.

It follows that in order to determine whether two 2-cells in $\Pi_2(\mathbf{Ab}_{fg})$ are equivalent it is sufficient to consider all chains of direct equivalences of length two or less.

References

[1] J. Bénabou, Introduction to bicategories, in *Reports of the Midwest Category Seminar*, LNM 40, Springer Verlag, New York, 1967, p. 1-77.

[2] R. J. MacG. Dawson, R. Paré, D. A. Pronk, The free adjoint construction, preprint.

[3] J. Lambek, How to program an infinite abacus, *Canad. Math. Bull.* 4 (1961), p. 295-302.

[4] S. Mac Lane, *Categories for the Working Mathematician*, 2nd ed'n, Springer Verlag, New York, 1997.

[5] M. L. Minsky, Recursive unsolvability of Post's problem of 'tag' and other topics in the theory of Turing machines, *Annals of Math.* 74 (1961), p. 437-455.

[6] S. Schanuel, R. Street, The free adjunction, *Cahiers Topologie Géom. Differentielle Catégoriques* 27 (1986), p. 81-83.