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Characterizing Tileorders

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Abstract. We define a tileorder to be a double order generated by a dissection of a rectangle into subrectangles. These structures are of interest both geometrically and as the order structures underlying double categories. We here give three different characterizations of those double orders which are tileorders.

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1. Introduction

A *double order* is a set T with two partial orderings, $\prec$ ('below') and $<$ ('to the left of'). A *tileorder* is a double order induced by a dissection of a rectangle into finitely many subrectangles. The objects of the double order are the subrectangles. $A \leqslant B$ if there exists a chain of tiles $A = X_1, X_2, \ldots, X_n = B$ such that the right edge of X_i intersects the left edge of X_{i+1} in more than one point; and $A \preceq B$ if there exists such a chain in which the top edge of each X_i intersects the bottom edge of X_{i+1} in more than one point.

For instance, in Figure 1, we have $a < b$, $a < c < d$, and $e < d$; while in the vertical direction, we have $a \succ e$, $b \succ c \succ e$, and $b \succ e$. However, not every

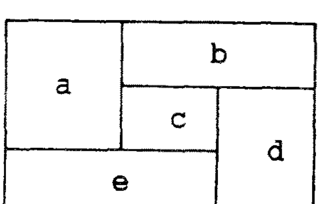


Fig. 1.

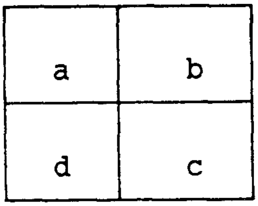


Fig. 2.

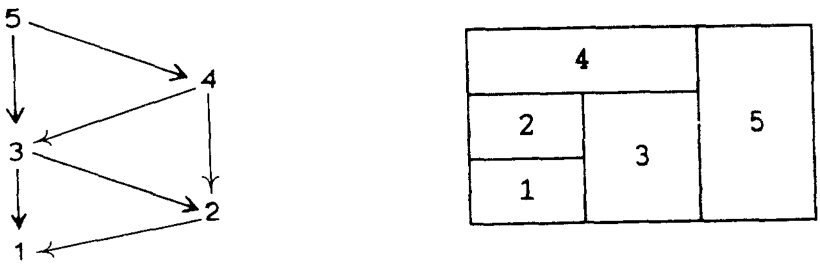


Fig. 3.

double order can be so represented. We list some illustrative examples:

• If A and B are rectangles, and the right edge of A intersects the left edge of B, then a translation of A to the right causes it to collide with B. Thus, the orderings $\leqslant$ and $\preceq$ are examples of *one-directional collision relations*. Rival and Urrutia have shown ([6], Theorem 1) that an order relation can be represented as a one-directional collision relation if and only if it is a truncated planar lattice.

Thus, for $< T, \preceq, \leqslant >$ to be a tileorder, it is clearly necessary that both $< T, \preceq >$ and $< T, \leqslant >$ be truncated planar lattices. However, the next example shows that this condition is not sufficient.

• The 2-element double order with $a < b$, $a \succ b$ cannot be represented as a dissection of a rectangle, as the two subrectangles would have to have two edges in common.

• The 3-element double order in which $a < b$, $c \prec b$, but a is not related to c by either $>$ or $\succ$ cannot be represented as a dissection of a rectangle. However, if a fourth element d is added so that $d < c$, $d \prec a$, the new double order can be realised (see Figure 2):

• Consider the following double order on $[1, 2, \ldots, n]$. $m < n$ if m is strictly less than n and n is even; and $m \prec n$ if m is strictly less then n and n is odd (Figure 3). This is a tileorder, as illustrated. However, if we reverse the order of 2 and 1, the resulting double order is not a tileorder (although its constituent orders are both truncated planar lattices, see Figure 4). The purpose of this paper is to characterize tileorders, and determine some of their properties. We will first exhibit a simple inductive construction, by which any tileorder can be built up, starting with eight elementary tileorders. This will be used to derive another characterization,

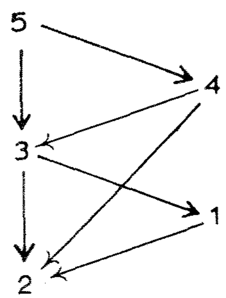


Fig. 4.

showing that a double order is a tileorder if and only if the maximal chains of its two component orders have certain intersection properties.

Finally, we derive a third characterization of tileorders, which is stated entirely in terms of 'small' configurations. Certain configurations (such as $a < b \prec a$) are forbidden; and others, such as $a < b \prec c$, must extend to larger configurations (in the example shown, we must also have $a \prec d < c$.) These conditions are shown to be necessary and sufficient.

While this particular problem does not seem to have been considered before, there is a moderately extensive literature on dissections of rectangles into rectangles. The majority of these papers (notably, in order of publication, [4], [1], [5]) have dealt with dissections of squares and rectangles into squares of different sizes, a combinatorial problem now susceptible to computational solutions via a well-known parallel with electrical circuit theory. More general dissections of rectangles into rectangles are dealt with in [2], in which some interesting results are derived on existence and average tile size of 'simple' tiled rectangles.

2. Regularity and Welding

The theory of DO's has several dualities; we may invert one or both of the constituent orders, or exchange them. Theorems and definitions will, for brevity, often be stated in one form only, understood to include all duals; for the details of these dualisations, we rely on the reader's common sense.

The two orders constituting a DO may be represented in terms of sets such as $\downarrow a = \{x : x \preceq a\}$ and $\leftarrow a = \{x : x \leqslant a\}$. We will call $< T', \prec', \leqslant' >$ a *sub-DO of* $< T, \prec, \leqslant >$ if $T' \subseteq T$, $a \prec' b \Rightarrow a \prec b$, and $a \leqslant' b \Rightarrow a \leqslant b$. If $a < b$ and there is no c such that $a < c < b$, we will say that a is a *left neighbour* or $<$*-neighbour* of b, and write $a <!b$.

We will call a DO *strongly antisymmetric* if, for $a \neq b$, at most one of $a < b$, $a > b$, $a \prec b$, $a \succ b$ hold. In any DO, we can define the following additional relations: $a \preceq\leqslant b$ if there exists c with $a \preceq c \leqslant b$, $a \leqslant\preceq b$ if there exists c with $a \leqslant c \preceq b$, $a \preceq\geqslant b$ if there exists c with $a \preceq c \geqslant b$, and $a \geqslant\preceq b$ if there exists c

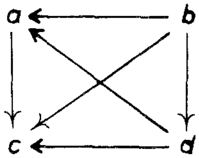


Fig. 5.

with $a \geqslant c \preceq b$. A DO is *rectangular* if $a \preceq\leqslant b \Leftrightarrow a \leqslant\preceq b$ and $a \preceq\leqslant b \Leftrightarrow a \leqslant\preceq b$; and *total* if, for every a, b, at least one of $a \preceq\leqslant b$, $a \succeq\leqslant b$, $a \preceq\geqslant b$, or $a \succeq\geqslant b$ holds. A DO which has all three of these properties will be called *regular*. In a regular double order $\preceq\leqslant$ and $\preceq\geqslant$ are themselves orders. We will write $\swarrow a$ for $\{x : x \preceq\leqslant a\}$, etc.

It will be shown below that all tileorders are regular. However, not every regular double order is a tileorder. The double order of Figure 5 is regular, but (by Theorem 1 or Theorem 2 below) is not a tileorder. A DO which has the property that the $\leqslant$-maximal elements and the $\leqslant$-minimal elements form a maximal $\prec$-chain, while the $\preceq$-maximal and $\preceq$-minimal elements form a maximal $<$-chain, will be called *well-bordered*. A well-bordered DO may be thought of as possessing a top, bottom, left, and right edge, each linearly ordered. Note also that a finite well-bordered DO has a unique $\preceq\leqslant$-maximal element, a unique $\preceq\geqslant$-maximal element, a unique $\succeq\leqslant$-maximal element and a unique $\succeq\geqslant$-maximal element; these may be thought of as the 'corners' of the DO. Tileorders are obviously well-bordered. Furthermore:

THEOREM 1. *A finite regular DO is well-bordered.*

Proof. Let a, b be two $\prec$-maximal elements of T. Then, w.l.o.g., there exists c such that $a \preceq c \leqslant b$ (note that if $a \succeq c \leqslant b$, then there exists d such that $a \leqslant d \preceq b$). As a is $\preceq$-maximal, we conclude that $a = c \leqslant b$; thus $\preceq$-max(T) is a $<$-chain. To show that this chain is maximal, suppose c, e to be $<$-neighbours in $\preceq$-max(T) but not in T. Let d be any element of T with $c < d < e$, and d' any $\preceq$-maximal element of T such that $d \prec d'$. But, as $c <!e$ in $\preceq$-max(T), either $d' < c < d$ or $d < e < d'$, contradicting our assumption of strong antisymmetry. □

If $< S, \preceq', \leqslant' >$ and $< T, \preceq, \leqslant >$ are well-bordered, and there is an order isomorphism $j : <' - \max(S) \to < - \min(T)$, we define their *(horizontal) weld*, $< S, \preceq', \leqslant' > \mathbin{+\!\!\!+} < T, \preceq, \leqslant >$ to be the DO with underlying set $S \cup T'/s \approx js$, in which a is a left (right, upper, or lower) neighbour of b in $< S \cup T, \preceq'', \leqslant'' >$ if and only if it is a left (right, upper, or lower) neighbour of b in $< S, \preceq', \leqslant' >$ or $< T, \preceq, \leqslant >$. Thus, $a \preceq'' b$ if $a \preceq' b$, or $a \preceq b$, or $a \preceq' s$ and $js \preceq b$, or $a \preceq js$ and $s \preceq' b$, and $a \leqslant'' b$ if $a \leqslant' b$, $a \leqslant b$, or $a \leqslant' s$ and $js \leqslant b$.

When S and T are well-bordered and there is an order isomorphism $j : \prec' - \max(S) \to \prec - \min(T)$, we define their *(vertical) weld* $< S, \preceq', \leqslant' > \# < T, \preceq, \leqslant >$ similarly, identifying the top boundary of S with the bottom boundary of T and taking the transitive closure. Intuitively, the weld operations join double orders

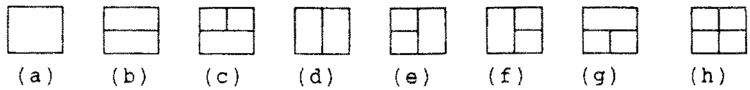


Fig. 6.

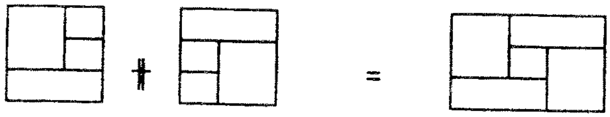


Fig. 7.

by identifying (say) the top boundary of one with the bottom boundary of the other. Note that the weld of two well-bordered DO's is itself well-bordered.

THEOREM 2. *The tileorders are precisely the double orders generated from the eight tileorders of Figure* 6 *by the weld operations.*

Proof. If two DO's can be realised as tileorders, then their weld can also be realised, by joining the two tiled rectangles along the appropriate edge and erasing the edges between them (Figure 7). It suffices, then, to show that any tileorder other than the eight above is the weld of two tileorders with strictly fewer tiles. It is easy to verify that any tiling other than the eight shown above contains two parallel but noncollinear internal edges. Then a line parallel to these edges and between them divides the tiling into two strictly smaller tilings, whose weld is the original one. □

Remark for categorists. Theorem 2 may be interpreted as characterizing tileorders as the cell elements of a certain double category. This double category has one object $*$, and its categories of horizontal and vertical morphisms are each freely generated by one element, h and v respectively. Thus, each is isomorphic to the natural number category $\mathbb{N}$. The identity cell on $*$ corresponds to the tileorder (6a), while the identity cells on h and v correspond to (6d) and (6b) respectively. Identity cells on the other horizontal morphisms correspond to welds of (6d) with itself; those on the other vertical morphisms correspond to welds of (6b) with itself.

The cells of the double category are generated by five cells corresponding to (6c),(6e),(6f),(6g), and (6h). The domains and codomains are determined by the number of tiles on the corresponding edge. For instance, the cell corresponding to tileorder (6c) has vertical domain h, vertical codomain equal to the horizontal identity, and horizontal domain and codomain both equal to v. The domains and codomains of (6e),(6f), and (6g) are defined analogously. The vertical domain and codomain of the cell corresponding to (6h) are both h, while the horizontal domain and codomain are both v. Vertical and horizontal composition of cells correspond to vertical and horizontal welding respectively. The cells of the free double category generated by these five cells and two morphisms correspond naturally to the tileorders. We will take this point up in more depth in an upcoming paper.

3. Maximal Chain Properties

A DO will be said to have the *$\prec$-parallel maximal chain* property ($\prec$-PMCP) if, given any two maximal $\prec$-chains K and L with $k_1 \prec k_2 \in K$, $l_1, l_2 \in L$, $k_1 \leqslant l_1$, $l_2 \leqslant k_2$, there exists $e \in K \cap L$ with $k_1, l_1 \prec e \prec k_2, l_2$. It will be said to have em orthogonal maximum chain property (OMCP) if every maximal $<$-chain intersects every maximal $\preceq$-chain precisely once.

THEOREM 3. *A DO with the OMCP is strongly antisymmetric and total; a DO with the OMCP and the PMCP is regular.*

Proof. If $a \preceq b$, $a \leqslant b$, but $a \neq b$, then a vertical chain through a and b intersects a horizontal chain through a and b in more than one element; this is forbidden by the OMCP.

Any maximal vertical chain through an element a intersects any maximal horizontal chain through an element b; thus $a \preceq\leqslant b$, $a \preceq\geqslant b$, $a \succeq\leqslant b$, or $a \succeq\geqslant b$, and any DO with the OMCP is total.

Let the DO also have the PMCP, and suppose, w.l.o.g., that $a \preceq c \leqslant b$. Exchanging $\leqslant$ and $\preceq$ in the previous argument, we also have $a \leqslant\preceq b$, $a \geqslant\preceq b$, $a \leqslant\succeq b$, or $a \geqslant\succeq b$. Suppose that $a \geqslant d \preceq b$. Then by the PMCP there exists e such that $a \preceq e \preceq c$, $d \preceq e \preceq b$; and $a \preceq b$. Similar arguments show that if $a \leqslant\succeq b$, then $a \leqslant b$; and if $a \geqslant\succeq b$, then $a = b$. Thus, the DO is rectangular. □

THEOREM 4. *A double order is a tileorder if and only if it has the OMCP and PMCP.*

Proof. To show that a tileorder has the OMCP, consider the *trail* of a maximal $\preceq$-chain $X_1 \prec X_2 \prec \cdots \prec X_m$ of rectangles, defined to be the polygonal arc with vertices $(x_i : 0 \leqslant i \leqslant m)$ where x_0 is the midpoint of the bottom face of X_1, x_m is the midpoint of the top face of X_m, and for $0 < i < m$, x_i is the midpoint of $X_i \cap X_{i+1}$. The trail is a connected path joining the top and bottom edges of the rectangle. Defining the trail of a maximal $\leqslant$-chain $Y_0 < Y_1 < \cdots < Y_n$ analogously, it is clear that they must intersect, and that the rectangle containing their point of intersection is in both chains. It remains to show that the chains do not intersect twice. Suppose, for a contradiction, that they do. Let $X_a = Y_b$ and $X_c = Y_d$ be in both chains, and (without loss of generality) assume that $a < c$, $b < d$, and that no X_i, $a < i < c$, is in both chains. Then x_a is above the trail of (Y_i), and as each successive x_j has a vertical coordinate greater than that of x_{j-1} it follows that $x_{a+1}, x_{a+2}, \ldots, x_{c-1}$ are also above the trail of (Y_i). But x_{c-1} is on the bottom edge of X_c so X_c cannot equal Y_d.

It remains to show that any DO $(X, \leqslant, \preceq)$ with both maximal chain properties may be realised as a tiling. This is done inductively. Suppose that such a DO contains a maximal $\prec$-chain K and elements $x, y \notin K$ with $x < k \in K$, $y > k' \in K$. Define $X_L = \{a : a \leqslant k \in K\}$, $X_R = \{a : a \geqslant k \in K\}$. By the OMCP, any maximal $<$-chain L through any element $a \in X$ intersects K; thus $X = X_L \cup X_R$. Furthermore, L only intersects K once; so $K = X_L \cap X_R$. X_L and X_R both have strictly fewer elements than X; in particular, $x \notin X_R$ and $y \notin X_L$. Furthermore, the sub-DPO's $(X_L, \leqslant, \preceq)$ and $(X_R, \leqslant, \preceq)$ also have the OMCP and the PMCP . To check this, we must consider three cases: two parallel $\preceq$-chains, two parallel $\leqslant$-chains, and two orthogonal chains.

The first case follows trivially, as maximal $\preceq$-chains of X_L are also maximal $\preceq$-chains of X. In the second and third cases, the maximal $\leqslant$-chains of X_L can be extended to maximal $\leqslant$-chains of X. The maximal chain properties for X require an intersection to exist. In the orthogonal case, the element at which the intersection occurs is an element of a maximal $\preceq$-chain of X_L, hence in X_L; in the $\leqslant$-parallel case, the intersection is required to occur to the left of a specified element of X_L, hence in X_L.

Furthermore, $X = X_L \nparallel X_R$. To show this, we need to verify that the relations $\leqslant$ and $\preceq$ in L can be recovered from those in X_L and X_R. If x and y are both in X_L or both in X_R this is trivial. Otherwise, suppose (without loss of generality) that $x \in X_L$, $y \in X_R$. Then if $x \leqslant$y, there is a maximal $\leqslant$-chain in X through x and y; by the OMCP this intersects K in an element w. As $w \in K$, $x \leqslant w$ in X_L and $w \leqslant y$ in X_R; thus $x \leqslant y$ in $X_L \nparallel X_R$. Similarly, if $x \preceq y$, there is a maximal $\preceq$-chain L through x and y, and there exist $k, k' \in K$ such that $x \leqslant k$, $k' \leqslant y$. Therefore, by the PMCP, K intersects L in some element w with $x \preceq w$ in X_L and $w \preceq y$ in X_R and $x \preceq y$ in $X_L \nparallel X_R$.

Thus, every finite DO with the maximal chain properties can be obtained by welding weld-irreducible DO's with the maximal chain properties and with no two elements separated by a maximal chain. By Theorems 1 and 3, such a weld-irreducible DO is well-bordered. Each of the four edges must be a one- or a two-element chain, and inspection shows that there are exactly eight possibilities.

If the bottom edge has one element, either it has no upper neighbours (case 1), one upper neighbour (case 2) or two upper neighbours (case 3). In each case, this exhausts the elements of the DO, as no vertical chain is of length 3. If the bottom edge has two elements, either nothing is above either of them (case 4), one element is above the left bottom element (case 5), one element is above the right bottom element (case 6), a common element is above both bottom elements (case 7) or each bottom element has a different top element above it (case 8).

In cases 4,5,6, and 7 the listed elements clearly exhaust those of the DO. In case 8, it remains to show that if $a > b \prec c < d$, $a \not> c$ and $d \not> b$. Suppose $a > c$; then $(a > c)$ is a maximal chain. Clearly $b \prec c$, and by the OMCP, a maximal $\prec$-chain through d meets this chain. Strong antisymmetry forbids $c \prec d$; and if $a \prec d$, then the chain $(a > c)$ separates b and d. Thus we conclude that in case 8, $a \not> c$ and $d \not> b$; another application of the OMCP allows us to conclude that $a \prec d$. Thus cases 1–8 correspond to the eight weld-irreducible tileorders of Figure 6, in the

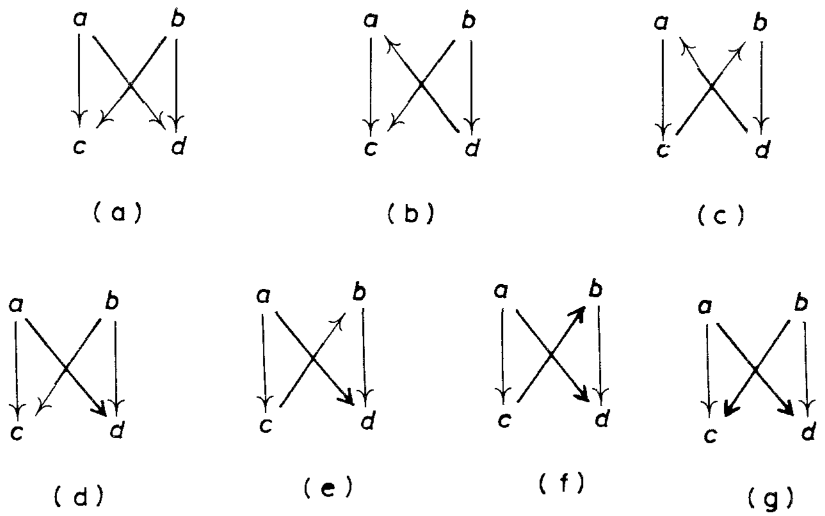


Fig. 8.

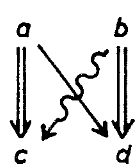
order shown. □

COROLLARY. *Tileorders are regular.*

4. Butterfly Factorization and Related Properties

The characterizations of tileorders given by Theorems 2 and 4 are intuitive, but tell us little about the behaviour of small sets of tiles. In the rest of this paper, we will investigate various other properties of tileorders which give rise to a rather surprising characterization, and are useful in the study of tileordered double categories.

Any diagram of the form



where the arrows $a \Rightarrow c$, $b \Rightarrow d$ represent the same one of $\{\leqslant, \geqslant, \preceq, \succeq\}$ and the arrows $a \to d$, $b \rightsquigarrow c$ each represent any of $\{\leqslant, \geqslant, \preceq, \succeq\}$ will be called a *butterfly diagram*. A butterfly diagram in a DO will be said to *factorize* if there exists an element e such that $a \to e \to d$, $b \rightsquigarrow e \rightsquigarrow c$. The main result of this section is that a finite regular double order is a tileorder if and only if all butterflies factorize.

Up to duality, every butterfly diagram is of one of the types in Figure 8 (note that the first three may also exist in an order). It is clear that (8b), or any butterfly equivalent under duality, factorizes trivially in any order. (8c), (8e), and (8g), in

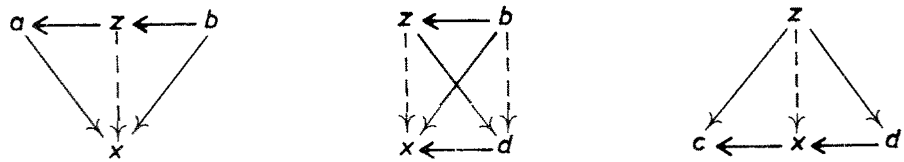


Fig. 9.

contrast, cannot occur in a regular DO except in the degenerate case in which $a = b = c = d$. This follows immediately from strong antisymmetry for (8c) and (8e); in (8g), rectangularity implies the existence of an element f with $c \geqslant f$, $d \succeq f$. Thus $b \geqslant f$ and $b \succeq f$; by strong antisymmetry, $b = f = c = d = a$.

The remaining three cases are of more interest. We will (8a) a $\preceq$*-homogeneous* butterfly; a $\leqslant$-homogeneous butterfly is defined dually, and a homogeneous butterfly is one which is either $\preceq$- or $\leqslant$-homogeneous. (8d), and the butterfly obtained from it by reversing the inequality $a \leqslant d$, will be called $\preceq$*-orthogonal* butterflies. Finally, (8f) will be called a $\leqslant$-parallel butterfly; and $\leqslant$-orthogonal, $\preceq$*-parallel*, orthogonal, and parallel butterflies are defined analogously. (The apparent inconsistency in the nomenclature for parallel butterflies makes the statements of various theorems below more consistent.)

If every $\preceq$-homogeneous (resp. $\leqslant$-homogeneous, homogeneous, orthogonal ...) butterfly in a DO factors, we will say that that DO has the $\preceq$*-homogeneous* (resp. $\leqslant$*-homogeneous, homogeneous, orthogonal...) butterfly factorization property*. These properties will generally be abbreviated, as the $\preceq$-HBFP, $\leqslant$-HBFP, HBFP, etc.

A DO will be called *convex* if the following and its duals hold:

$$x \preceq a \leqslant z, \quad x \preceq b \geqslant z \Rightarrow x \preceq z. \tag{1}$$

It is clear that (1) is its own dual upon inverting $\leqslant$; a moment's reflection will show that, given rectangularity, the existence of the dashed arrows in diagrams of the three types in Figure 9 are equivalent, and so (1) is also self-dual upon inverting $\preceq$. We may thus, for rectangular DOs, reduce convexity to two subaxioms, $\preceq$-convexity (illustrated) and $\leqslant$-convexity. Note that a DO $< T, \preceq, \leqslant >$ is $\preceq$-convex if and only if $\searrow a \cap \nearrow a = \uparrow a$. This motivates the following result:

THEOREM 5. *Let $a \preceq' b$ if $a \preceq\leqslant b$ and $a \preceq\geqslant b$. Then if $< T, \preceq, \leqslant >$ is a regular DO, $< T, \preceq', \leqslant >$ is a $\preceq'$-convex regular DO and $< T, \preceq', \leqslant' >$ is a convex regular DO.*

Proof. We shall show that $< T, \preceq', \leqslant' >$ is regular and convex; the proof for $< T, \preceq', \leqslant >$ is similar and slightly simpler. Totality follows immediately from that of $< T, \preceq, \leqslant >$. Given that $a \preceq' b \leqslant' c$, see the result in Figure 10.

By rectangularity, there exists $h \in T$ such that $e \succeq h \leqslant f$. Thus $a \preceq h \leqslant c$, and so there exists i with $a \leqslant i \preceq c$; *a fortiori*, $a \preceq' i \preceq' c$; so $< T, \preceq', \leqslant >$ is rectangular. Replacing c by a in the above diagram, a similar argument shows that $< T, \preceq', \leqslant>$ is strongly antisymmetric.

(**Remarks for categorists.** If we define a morphism of double orders to be a function preserving both $\preceq$ and $\leqslant$, the last two constructions have nice categorical properties. First, both horizontal and vertical welds are easily seen to be a special type of pushout in the category of DO's. Moreover, if a DO $< T, \preceq, \leqslant>$ has a convex extension $< T, \preceq', \leqslant'>$, the new relations on T created by the convexification process already exist in T'; so this construction is reflective.)

The next result must surely be well-known. Its proof is easy:

PROPOSITION. *In a finite order, the following are equivalent:*

(i) $(\forall a, b)\ ((\downarrow a \cap \downarrow b = \emptyset)$ or $(\exists e)(\downarrow a \cap \downarrow b = \downarrow e))$;

(ii) $(\forall c, d)\ ((\uparrow c \cap \uparrow d = \emptyset)$ or $(\exists e)(\uparrow c \cap \uparrow d = \uparrow e))$;

(iii) *The order has the* $\preceq$-HBFP.

Proof. Trivially, (i) $\Rightarrow$ (iii). To show the converse, suppose that $\downarrow a \cap \downarrow b$ is nonempty; then, being finite, it has at least one maximal element. The factorization (iii) implies that this maximal element must be unique, as if $c, d \in \downarrow a \cap \downarrow b$, either $c \succeq d$, $d \succeq c$, or there exists an element $e \succ c, d$; thus (iii) $\Rightarrow$ (i). Dually, (ii) $\Rightarrow$ (iii). $\square$

Such an order will be called a *partial lattice*. The next theorem establishes the relationship between partial lattices and some related structures.

THEOREM 6. *The following implications exist and are irreversible.*

$$\begin{array}{ccc} \textit{truncated planar lattice} & & \textit{lattice} \\ & \Downarrow \quad \Downarrow & \\ & \textit{partial lattice} & \\ & \Downarrow & \\ & \textit{truncated lattice} & \end{array}$$

Proof. It is clear that any lattice is a partial lattice, and any partial lattice may be made into a lattice by adding a top element and a bottom element. To show that any truncated planar lattice is a partial lattice, consider the two essentially different ways of embedding (8a) in a planar lattice diagram (Figure 11).
In case (11a), the factorization is obvious. In case (11b), $c \succeq d$, and (8a) factors through c. For suppose not; then $c \wedge d$ is strictly below d, and the arrow $c \succeq (c \wedge d)$ must cross (WLOG) $b \succeq d$ in a node x with $c \succeq x \succeq d$, a contradiction.

Finally, none of these implications is reversible. (8a) is a truncated lattice which is not a partial lattice. (11a) is a partial lattice which is not a lattice; and the cubic lattice $\mathbf{2}^3$ is a partial lattice but not planar. $\square$

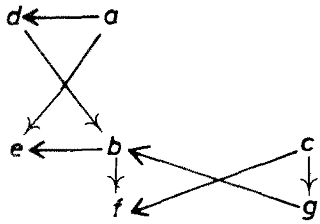


Fig. 10.

(a) (b)

Fig. 11.

This, in combination with Theorem 1 of [6], shows that every tileorder has the HBFP. However, we can also prove this indirectly. Theorem 8, below, states that any DO with the PBFP and the OBFP has the HBFP; and we shall see that every tileorder does, in fact, have the former two properties.

THEOREM 7. *For a finite regular DO, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)*

(i) $< T, \preceq, \leqslant >$ *is* $\preceq$*-convex and* $< T, \preceq >$ *is a partial lattice;*

(ii) *The DO has the* $\preceq$*-PBFP*

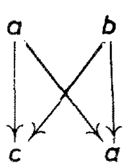
(iii) $< T, \preceq, \leqslant >$ *is* $\preceq$*-convex.*

Proof. (i) $\Rightarrow$ (ii): By convexity, $a \preceq b$ and $c \preceq d$ in (8f); thus the factorization follows by the previous proposition.

(ii) $\Rightarrow$ (iii): If (8f) factors, then $a \preceq e \preceq b, c \preceq e \preceq d$. □

THEOREM 8. *If a regular DO* $< T, \preceq, \leqslant >$ *has the* $\preceq$*-OBFP and the* $\preceq$*-PBFP, then* $< T, \preceq >$ *is a partial lattice.*

Proof. Suppose that



By totality, we have (up to duality) one of the two configurations of Figure 12.

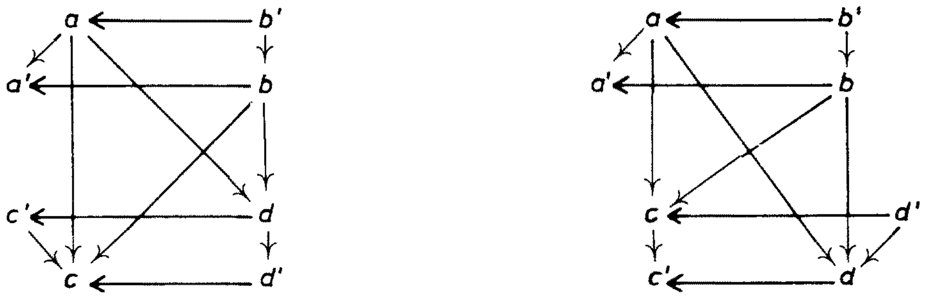


Fig. 12.

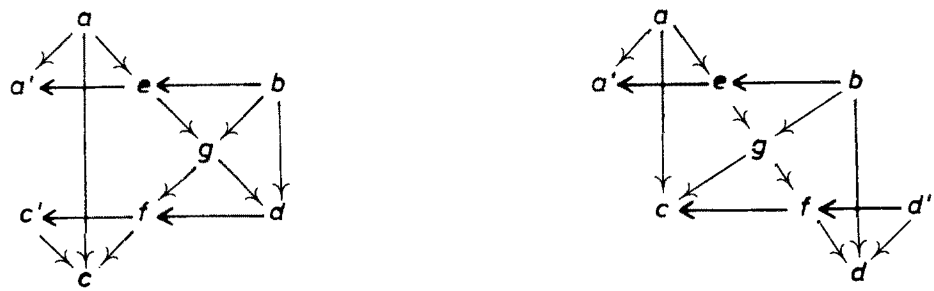


Fig. 13.

In the left-hand diagram, $a \succeq a' \leqslant b \succeq d \preceq a$, so by the $\preceq$-OBFP there exists e with $a \succeq e \succeq d$, $a' \leqslant e \leqslant b$. Similarly, there exists f with $b \succeq f \succeq c$, $c' \leqslant f \leqslant d$. But then $e \leqslant b \succeq f \leqslant d \preceq e$, so by the $\preceq$-PBFP there exists g with $e \succeq g \succeq d$, $b \succeq g \succeq f$. But then $a \succeq g \succeq c$, so the original homogeneous butterfly diagram factors (Figure 13). A similar argument yields g in the right-hand diagram of Figure 12.

THEOREM 9. (i) *A DO with the $<$-PMCP has the $\leqslant$-PBFP;* (ii) *A convex DO with the OMCP has the OBFP.*

Proof. (i) follows trivially from the definition of the $<$-PMCP if we extend the relations $a < d$, $c < b$ of (8f) to maximal $<$-chains. To prove (ii), extend the relations $a < d$, $b \succ c$ of (8d) to maximal chains. These must intersect in some element e; it suffices to show that $a < e < d$, $b \succ e \succ c$. Suppose otherwise; then, without loss of generality, we have one of the following cases:

$d < e$, $b \prec e$: The OMCP implies strong antisymmetry; but (8d) has $d \prec b$, and hence $d \prec e$.

$d < e$, $c \prec e \prec b$: Then $c \prec e > d$, and (8d) also has $c \prec a < d$. Thus by convexity $c \prec d$, and (8d) factors through d.

$e \prec c$, $a < e < d$: (8d) has $c \prec a$, which would imply $e \prec a$, ruled out by strong antisymmetry. □

COROLLARY. *Every tileorder is convex.*

The OMCP on its own does not imply either form of the OBFP , even for regular DO's, as shown by the following counterexample, in which all orthogonal pairs of maximal chains intersect but the orthogonal butterfly with vertices a, b, c, d does not factorize (Figure 14).

A DO will be said to have the $\prec$-neighbour chain property if the set $\{x : x \prec!a\}$ of lower neighbours of any element a and the set $\{x : x \succ!a\}$ of upper neighbours of any element a are totally $\leqslant$-ordered. Interchanging $\preceq$ and $\leqslant$ yields the dual $<$*-neighbour chain property*; a DO with both has the *neighbour chain property*.

It will be shown below that every tileorder has the neighbour chain property. This fact is used in [3] to show that certain operations are well-defined in double categories. The next two theorems indicate how the neighbour chain property is related to previously introduced properties.

THEOREM 10. *A finite regular DO with the $\prec$-NCP and the $\succeq$-PBFP has the $\prec$-PMCP.*

Proof. Suppose K, L to be maximal $\prec$-chains, with $k_1 \prec k_2 \in K$, $l_1, l_2 \in L$, such that $k_1 \leqslant l_1$, $k_2 \geqslant l_2$. The DO is chain-trichotomic, so that for every element $k \in K$ there exists $l \in L$ with $k > l$, $k = l$, or $k < l$. Thus, there must exist $k, k' \in K, l, l' \in L$ with $k_1 \preceq k \prec!k' \preceq k_2, k \leqslant l$, and $k' \geqslant l'$. Strong antisymmetry forbids $l \succeq l'$, so $l \prec l' \leqslant k' \succeq k \leqslant l$. This $\preceq$-parallel butterfly factorizes, with $l \preceq e \preceq l', k \preceq e \preceq k'$. But as $k \prec!k'$, we have (without loss of generality) $e = k$, whence by strong antisymmetry $l = k$. □

Open question: We do not at present know precisely how strong the PBFP is for finite regular DO's. Every example of a regular finite DO with the PBFP that we have been able to construct has also been a partial lattice and had the PMCP. This suggests the (perhaps foolhardy) conjecture that every finite DO with the PBFP has these properties. If this were so, clearly several of the theorems presented here could be formally strengthened.

THEOREM 11. *A DO with the $\preceq$-OBFP has the $\prec$-neighbour chain property.*

Proof. Let c, d be immediately below a. By totality, there exists (without loss of generality) b such that $c \leqslant b \preceq d$. Then, by hypothesis, there exists e such that $c \leqslant e \leqslant b, d \preceq e \preceq a$. If $e = a$, then $a \leqslant c$, and by strong antisymmetry $a = c$, which contradicts our original assumption. Thus $e \neq a$; but $d \prec!a$, so $d = e \geqslant c$.□

COROLLARY. *A finite regular DO with the $\preceq$-OBFP and the $\succeq$-PBFP has the $\prec$-PMCP.*

Note that the converse to Theorem 11 is not true: Figure 14 above shows a DO that has the NCP but contains an orthogonal butterfly that does not factor.

A condition closely related to the neighbour chain property is *$\prec$-chain trichotomy.* A DO has this property if, given any maximal $\prec$-chain K and any element a, exactly one of $\{a \in K, a < k \in K, a > k \in K\}$ holds. *$<$-chain trichotomy* is defined similarly, interchanging $\preceq$ and $<$; and a DO with both dual forms of this property is *chain-trichotomic.* This property is important, as a DO that has it is divided into two sub-DOs by any maximal chain, which intersect only in their boundaries; and it is the weld of those sub-DOs. All tileorders are chain-trichotomic; this will follow from Theorem 13.

THEOREM 12. *A regular, $\leqslant$-convex and $\prec$-chain trichotomic DO has the $\prec$-neighbour chain property.*

Proof. Let b, c be distinct elements atomically below a; by totality there exists (without loss of generality) d such that $b \leqslant d \succeq c$. There exists a maximal $\prec$-chain K through a and c, and by hypothesis there exists $k \in K$ with $b > k$ or $b < k$. Either $k \preceq c$ or $k \succeq a$; but in the latter case $k \succ b$, ruled out by strong antisymmetry; so $k \preceq c \preceq d$. If $k < b$, then $k < d$, again impossible by strong antisymmetry; we thus have $b < k \preceq c \preceq d > b$, and by $<$-convexity, $b < c$. □

THEOREM 13. *A finite, regular DO $< T, \preceq, \leqslant >$ is $\prec$-chain trichotomic if one of the following holds:*

(i) *it is $\leqslant$-convex, and has the $\prec$-NCP and the $\preceq$-PBFP;*

(ii) *it has the $\prec$-NCP and the PBFP;*

(iii) *it is convex, has the $\prec$-NCP, and $< T, \preceq >$ is a partial lattice;*

(iv) *it is convex and has the $\prec$-NCP and the $\leqslant$-OBFP;*

(v) *it is convex and has the OBFP.*

Proof. (ii) $\Rightarrow$ (i), (iii) $\Rightarrow$ (i), and (v) $\Rightarrow$ (iv). We shall prove cases (i) and (iv) in parallel, as much of the proofs overlap. Let $\{k_i : 0 \leqslant i \leqslant n\}$ be the elements of a maximal $\prec$-chain K. Any element $a \not\in K$ of the DO is related to each k_i by one of $\{\leqslant\preceq, \leqslant\succeq, \geqslant\preceq, \geqslant\succeq,\}$. If $k_i \prec k_{i+1}$ and $k_i \leqslant\succeq a$, then $k_{i+1} \leqslant\succeq a$; so $\nwarrow \boldsymbol{a} \cap \boldsymbol{K}$ **is** $\downarrow$**-closed in** $\boldsymbol{K}$**. Similarly,** $\nearrow \boldsymbol{a} \cap \boldsymbol{K}$ **is** $\uparrow$**-closed,** $\searrow \boldsymbol{a} \cap \boldsymbol{K}$ **is** $\downarrow$**-closed,** and $\swarrow a \cap K$ is $\downarrow$-closed in K. But the union of these four is all of K, while $\nearrow a \cap \swarrow a \cap K = \searrow a \cap \nwarrow a \cap K = \{a\} \cap K = \emptyset$. As $k_0 \in \searrow a \cup \swarrow a$ and $k_n \in \nearrow a \cup \nwarrow a$, it follows that there exist adjacent chain elements k_i, k_{i+1} with $k_i \in \searrow a \cup \swarrow a$ and $k_{i+1} \in \nearrow a \cup \nwarrow a$.

Let us first consider the case in which $k_i \leqslant\preceq a \preceq\leqslant k_{i+1}$ or $k_i \geqslant\preceq a \geqslant\preceq k_{i+1}$ (without loss of generality, assume the former).

(i) There exist p, q with $k_i \preceq p \leqslant a \leqslant q \preceq k_{i+1}$. If (say) $p = k_i$, then $a \geqslant k_i$ and we are done; so we may assume that $k_i \prec p$, $q \prec k_{i+1}$. Then there exist l, m such that $k_i \prec! l \preceq p, q \preceq m \prec! k_{i+1}$; by the $\preceq$-neighbour chain property, $l \geqslant k_{i+1}$ or $l \leqslant k_{i+1}$, and $m \geqslant k_i$ or $m \leqslant k_i$. If $l > k_{i+1}$ or $m < k_i$ (say the former), then $a \geqslant\succeq l > k_{i+1} \geqslant\succeq a$, impossible by strong antisymmetry; so $l \leqslant k_{i+1}$ and $m \geqslant k_i$. But if $l \leqslant k_{i+1} \succ m$ and $l \leqslant\preceq a \leqslant\preceq m$, by $\leqslant$-convexity $l \leqslant m$. But we also have $l \leqslant k_{i+1} \succ k_i \geqslant m$, so the resulting $\leqslant$-orthogonal butterfly factors through an element e with $l \leqslant e \leqslant m$, $k_i \prec e \prec k_{i+1}$. As $k_i \prec! k_{i+1}$, e = (without loss of generality) k_i. Then $l \leqslant k_i \prec l$, impossible by strong antisymmetry.

(iv) There exist r, s with $k_i \leqslant r \preceq a \preceq s \leqslant k_{i+1}$. Then k_i, r, s, and k_{i+1} form a $\preceq$-parallel butterfly, which factors through an element e with $r \preceq e \preceq s$, $k_i \preceq e \preceq k_{i+1}$. As $k_i \prec! k_{i+1}$, without loss of generality $e = k_{i+1}$, so by strong antisymmetry $s = k_{i+1}$, and $a \preceq k_{i+1}$. By finiteness there exists an element l such that $a \preceq l \prec! k_{i+1}$. By the $\prec$-NCP, $l \leqslant k_i$ or $l \geqslant k_i$. In the former case, $r \geqslant k_i \geqslant l \succeq a \succeq r$, so $l = r = k_i = a$. Otherwise, $a \succeq r \geqslant k_i$ and $a \preceq l \geqslant k_i$. By $\leqslant$-convexity, $a \geqslant k_i$.

We now turn to the remaining cases, in which (say) $k_i \leqslant\preceq a \geqslant\preceq k_{i+1}$. Then there exist p, q with $k_i \preceq p \leqslant a$, $k_{i+1} \succeq q \leqslant a$, and as above, we may assume that there exist l, m with $k_i \prec! l \preceq p$, $k_{i+1} \succ m \succeq q$, $l \geqslant k_{i+1}$ or $l \leqslant k_{i+1}$, and $m \geqslant k_i$ or $m \leqslant k_i$. If $l < k_{i+1}$ and $m < k_i$, we again consider (i) and (ii) separately.

(i) We have $l \leqslant k_{i+1} \succ m \leqslant k_i \prec l$, so, by $\preceq$-convexity, $m \prec l$ (see Figure 15).

Thus $q \preceq l$; we also know that $l \preceq p$, $p \leqslant a$, $q \leqslant a$, so by $\leqslant$-convexity $l \leqslant a$. Then l, a, m, and k_{i+1} form an $\leqslant$-orthogonal butterfly configuration, which factors through some e such that $m \preceq e \preceq k_{i+1}$, $l \leqslant e \leqslant a$. As $k_{i+1} \succ! m$, either $m = e$, in which case by strong antisymmetry $m = l$ and $k_i \preceq m = l \preceq k_{i+1}$, contradicting our original assumptions; or else $k_{i+1} = e \leqslant a$, and we are done.

(iv) We have $l \leqslant k_{i+1} \succ m \leqslant k_i \prec l$; this parallel butterfly diagram factors, and we get $k_i \preceq e \preceq l, m \preceq e \preceq k_{i+1}$. But as $k_i \prec! k_{i+1}$, without loss of generality $k_{i+1} = e \preceq l$. Thus by strong antisymmetry $k_{i+1} = l$; so $k_{i+1} \preceq p \leqslant a$. But $k_{i+1} \succeq\leqslant a$, so by convexity $k_{i+1} \leqslant a$.

Otherwise, without loss of generality, $k_{i+1} < l$. In that case, $a \geqslant p \succeq l > k_{i+1}$, $a \geqslant q \preceq k_{i+1}$, so that, by $\leqslant$-convexity, $a \geqslant k_{i+1}$. □

COROLLARY: *A finite regular DO in which parallel and orthogonal butterfly diagrams factorize has chain trichotomy and the neighbour-chain property.*

The following example shows that we cannot hope to weaken the conditions of Theorem 13 too much more; the DO exhibited in Figure 16 has the $\leqslant$-PBFP, $\prec$-NCP, the $\preceq$-OBFP, and both forms of convexity, but fails to be chain-trichotomic.

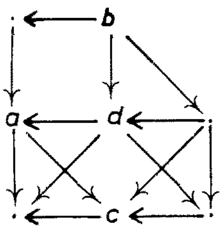


Fig. 14.

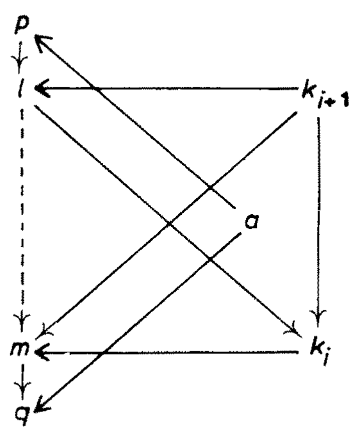


Fig. 15.

THEOREM 14. *A finite regular $\leqslant$-chain-trichotomic DO with the $\preceq$-OBFP has the OMCP.*

Proof. Let K be a maximal $\leqslant$-chain, and L a maximal $\prec$-chain. By $\leqslant$-chain trichotomy, every element of L is either above or below some element of K. Thus there must exist $l, l' \in L$ and $k, k' \in K$ such that $k \succeq l \preceq ! l' \succeq k'$. As (without loss of generality) $k \leqslant k'$, by the usual argument $l = l' = k = k'$; so the chains intersect. Strong antisymmetry ensures that they cannot intersect more than once. □

THEOREM 15. *For a finite DO, the following are equivalent:*

(i) *It is a tileorder;*

(ii) *It has the OMCP and the PMCP;*

(iii) *It is regular, and has the OBFP and PBFP;*

(iv) *It is regular, and all butterflies factorize.*

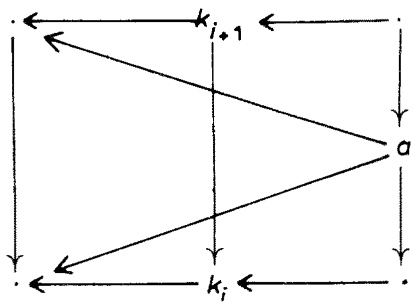


Fig. 16.

Proof. The equivalence of (i) and (ii) follows from Theorem 4, that of (iii) and (iv) from Theorem 8. By Theorems 7 and 9, (ii) $\Rightarrow$ (iii). If a DO has the OBFP and PBFP, by Theorem 11 it also has the NCP, whence by Theorem 10 it has the PMCP. Furthermore, Theorem 13 implies (in two distinct ways) that it has chain trichotomy, so by Theorem 14 it has the OMCP. Thus (iii)-(ii). □

5. Tileorder Recognition Algorithms

The characterization of a tileorder in Theorem 1, while simple, does not give an effective way to subdivide an abstract double order, and so does not in itself constitute an effective algorithm to recognize tileorders. (It does provide a recursive algorithm to generate, with some repetition, all tileorders of a given (small) size.)

However, as we have established that all tileorders are chain-trichotomic, it follows that any maximal chain in a tileorder, except the four 'edges', subdivides the tileorder into smaller pieces. This yields a polynomial time algorithm that determines whether an n-element double order is a tileorder.

We can, in $O(n^2)$ operations, construct such a chain (should it exist) in the given double order. Should it fail to exist, then either the double order is one of the primitive tileorders of Figure 4, or it is not a tileorder. We can, again with $O(n^2)$ operations, test the remaining elements to determine whether chain trichotomy holds with respect to that chain. If it does not, then the order is not a tileorder. If it does, then the double order may be effectively subdivided into smaller double orders, and it is a tileorder if and only if they are both tileorders. This recursion can require at most n repetitions, and can thus be carried out in $O(n^3)$ operations.

Theorem 4 provides, in theory, a recognition algorithm for tileorders; but as the number of maximal chains in a tileorder increases exponentially with its size, it is not efficient. Theorem 15 also suggests a recognition algorithm, in which every parallel and orthogonal butterfly configuration is tested to determine whether it factorizes. This, however, potentially requires $O(n^5)$ steps, and is therefore inefficient compared with that obtained using Theorem 1 and chain trichotomy. This can be tested in polynomial time, and thus constitutes a reasonably efficient recognition algorithm for tileorders.

6. Open Questions

The results above leave various questions open. For instance, it would be interesting to know what tileorders can be realised by dissections of squares into squares, of rectangles into squares, or of rectangles into rectangles of equal area. (It is easily shown that none of these representations is always possible!) While we can find upper and lower bounds fairly easily, determining the number $T(n)$ of tileorders with n elements appears to be an extremely difficult problem. The related, and probably even more difficult, problem of enumerating the dissections of a square into rectangles of equal area is problem 34 of [7].

Finally, the extension of these results into three dimensions appears fraught with complications; in particular, it is easily observed that strong antisymmetry, as defined here, does not hold for dissections of a rectangular 'brick' into 'sub-bricks'.

Acknowledgements

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