On Traces in Categories of Contractions

Extended Abstract

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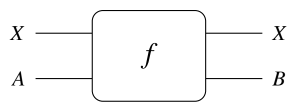
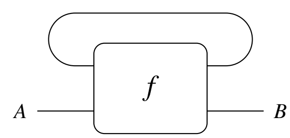
In memory of Phil Scott, 1947–2023

Traced monoidal categories are used to model processes that can feed their outputs back to their own inputs, abstracting iteration. The category of finite dimensional Hilbert spaces with the direct sum tensor is not traced. But surprisingly, in 2014, Bartha showed that the monoidal subcategory of isometries is traced. The same holds for coisometries, unitary maps, and contractions. This suggests the possibility of feeding outputs of quantum processes back to their own inputs, analogous to iteration. In this paper, we show that Bartha's result is not specifically tied to Hilbert spaces, but works in any dagger additive category with Moore-Penrose pseudoinverses (a natural dagger-categorical generalization of inverses).

1 Introduction

A trace on a symmetric monoidal category $(\mathbf{C}, \oplus)$ is an operation that assigns to each $f\colon A \oplus X \to B \oplus X$ another map $\mathrm{Tr}^X f\colon A \to B$, satisfying some well-known axioms [10, 25]. In string diagrams, traces are represented by looping an output of f back to the corresponding input, as in the following diagram.

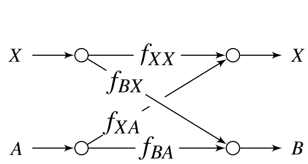
$f =$ 

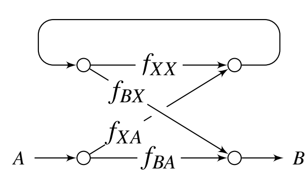
$\mathrm{Tr}^X f =$ 

In categories of vector spaces, there are two relevant monoidal structures: the "multiplicative" tensor $\otimes$ and the "additive" tensor $\oplus$, also known as biproduct or direct sum. The multiplicative tensor on finite-dimensional vector spaces has a well-known trace, induced by the compact closed structure. But in this paper, we are interested in the additive tensor.

A natural way to try to define an additive trace on a category of vector spaces is by the following sum-over-paths formula, which is motivated by the accompanying string diagrams.

$$f = \begin{matrix} & A & \oplus & X \\ B & & & \\ \oplus & & & \\ X & & & \end{matrix} \begin{pmatrix} f_{BA} & f_{BX} \\ f_{XA} & f_{XX} \end{pmatrix} \qquad \mathrm{Tr}^X f = f_{BA} + f_{BX} \circ f_{XA} + f_{BX} \circ f_{XX} \circ f_{XA} + f_{BX} \circ (f_{XX})^2 \circ f_{XA} + \cdots$$





The idea is similar to that of matrix multiplication, which can be formulated as a sum over all paths from a given input coordinate to a given output coordinate. However, the sum-over-paths formula does not define a total operation because the sum may not converge. Indeed, there is no totally defined trace with respect to $\oplus$ on any category of finite (or infinite) dimensional vector spaces [9].

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Therefore, it came as a surprise when Bartha showed in [3] that the category of finite dimensional Hilbert spaces and *isometries* has a well-defined additive trace. In particular, not only does Bartha's trace of an isometry always exist, but it is again an isometry. By duality, Bartha's trace also works for coisometries, and therefore also for unitary maps. Moreover, Andrés-Martínez pointed out that Bartha's trace further generalizes to all contractions [1]. These results suggest that there might be some physical interpretation of loops in quantum systems, but we do not know what it is.

In this paper, we show that Bartha's result is not specifically tied to Hilbert spaces, but works in any dagger additive category with suitable additional structure. The specific additional structure that we need to assume is the existence of Moore-Penrose pseudoinverses.

In a nutshell, a pseudoinverse of an arrow $f\colon A \to B$ is an arrow $f^\circ\colon B \to A$ such that both $f \circ f^\circ$ and $f^\circ \circ f$ are self-adjoint and $f \circ f^\circ \circ f = f$ and $f^\circ \circ f \circ f^\circ = f^\circ$. Pseudoinverses are unique when they exist, and they generalize inverses. Moreover, the definition of pseudoinverse is purely algebraic and makes sense in any dagger category [5].

Our main result is the following:

Theorem 1. *Given any dagger additive category with pseudoinverses, there is a totally defined trace on each of the following monoidal subcategories:*

- *the unitaries,*
- *the isometries,*
- *the coisometries, and*
- *the contractions.*

Moreover, in the cases of unitaries and contractions, which are dagger monoidal subcategories, the trace is a dagger trace.

After reviewing some background material in Section 2, we introduce contractions in Section 3 and pseudoinverses in Section 4, and prove some of their required properties. Section 5 is devoted to the proof of the main theorem.

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2 Background

2.1 Dagger categories

We recall some basic definitions and properties of dagger categories to fix the notation for the rest of the paper. For a more detailed treatment, see [23, 8, 11].

Definition 2.1 (Dagger category)**.** A *dagger category* is a category equipped with an identity-on-objects involutive contravariant functor, denoted $(-)^\dagger$. In other words, for $f\colon A \to B$, we have $f^\dagger\colon B \to A$, and we have the following properties:

- $f^{\dagger\dagger} = f$,
- $(1_A)^\dagger = 1_A$, and
- $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$.

For example, the category **Hilb** of Hilbert spaces and bounded linear maps is a dagger category. Its subcategory **FdHilb** of finite dimensional Hilbert spaces is also a dagger category.

Definition 2.2 (Properties of arrows)**.** An arrow $f\colon A \to B$ in a dagger category is called an *isometry* if $f^\dagger \circ f = 1_A$, a *coisometry* if $f \circ f^\dagger = 1_B$, and *unitary* if it is an isometry and a coisometry. Equivalently, f is unitary if it is invertible and $f^{-1} = f^\dagger$. An arrow $f\colon A \to A$ is *self-adjoint* (or *hermitian*) if $f = f^\dagger$.

In this paper, we use the symbol $\oplus$ to denote the monoidal product, because we are mainly interested in monoidal structures that are induced by biproducts.

Definition 2.3 (Dagger monoidal category)**.** A *dagger monoidal category* is a dagger category that is also monoidal, such that $(-)^\dagger$ is a strict monoidal functor. More explicitly, this means that the monoidal structure isomorphisms (i.e., associators and unitors) are unitary, and for all arrows f and g, we have

$$(f \oplus g)^\dagger = f^\dagger \oplus g^\dagger.$$

In a dagger (monoidal) category, the isometries, coisometries, and unitary maps each form a (monoidal) subcategory, i.e., they are closed under compositions (and monoidal products).

Definition 2.4 (Dagger finite biproduct category)**.** A *dagger finite biproduct category* is a dagger category that also has finite biproducts, such that the projection maps $\pi_i\colon A_1 \oplus A_2 \to A_i$ and the inclusion maps $\iota_i\colon A_i \to A_1 \oplus A_2$ satisfy $\pi_i = \iota_i^\dagger$.

The dagger biproducts of course also form a dagger monoidal structure. As usual in any category with finite biproducts, there is a zero object 0, and we can define the addition of arrows $f, g\colon A \to B$ in the usual way by $f + g = A \to A \oplus A \xrightarrow{f \oplus g} B \oplus B \to B$. There are also zero maps $0\colon A \to 0 \to B$. In this way, every (dagger) finite biproduct category is enriched over commutative monoids.

Definition 2.5 (Negatives, additive category)**.** We say that a (dagger) finite biproduct category *has negatives* if for every $f\colon A \to B$, there exists $-f\colon A \to B$ such that $f + (-f) = 0$. A (dagger) finite biproduct category with negatives is called a (dagger) *additive* category.

Definition 2.6 (Positive map)**.** A map $a\colon A \to A$ in a dagger category is *positive* if there exists $f\colon A \to B$ with $a = f^\dagger \circ f$.

Positive maps in **Hilb** are positive operators in the usual sense. Every positive map is self-adjoint, and the sum of positive maps is positive if we have dagger biproducts. Given two maps $f, g\colon A \to A$, we say that $f \le g$ if there exists some positive a such that $g = f + a$.

Lemma 2.7. *Let $f, g\colon A \to A$ and assume $f \le g$. (a) For all $h\colon A \to B$, we have $h \circ f \circ h^\dagger \le h \circ g \circ h^\dagger$. (b) For all $f', g'\colon A' \to A'$ with $f' \le g'$, we have $f \oplus f' \le g \oplus g'$.*

Proof. Since $g = f + a$ and $g' = f' + a'$ for some positive a and a', we have

$$h \circ g \circ h^\dagger = h \circ f \circ h^\dagger + h \circ a \circ h^\dagger \quad \text{and} \quad g \oplus g' = (f + a) \oplus (f' + a') = (f \oplus f') + (a \oplus a').$$

It is easy to see $h \circ a \circ h^\dagger$ and $a \oplus a'$ are positive, which implies both claims. □

2.2 Matrices

In a finite biproduct category, consider objects $A = A_1 \oplus \cdots \oplus A_m$ and $B = B_1 \oplus \cdots \oplus B_n$. It is well-known that a map $f\colon A \to B$ is in one-to-one correspondence with a matrix (f_{ji}), where $f_{ji}\colon A_i \to B_j$. We write

$$f \;=\; \begin{array}{c} \\ B_1 \\ \oplus \\ \vdots \\ \oplus \\ B_n \end{array} \!\!\begin{array}{c} \scriptstyle A_1 \;\oplus\; \cdots \;\oplus\; A_m \\ \begin{pmatrix} f_{11} & \cdots & f_{1m} \\ & & \\ \vdots & \ddots & \vdots \\ & & \\ f_{n1} & \cdots & f_{nm} \end{pmatrix} \end{array}.$$

Then composition of morphisms coincides with the usual formula for matrix multiplication, given by

$$(g \circ f)_{ki} = \sum_{j \in J} g_{kj} \circ f_{ji}.$$

In a dagger finite biproduct category, the dagger of a morphism coincides with the adjoint of its matrix:

$$(f^\dagger)_{ij} = (f_{ji})^\dagger.$$

Given a matrix

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} : A_1 \oplus A_2 \to B_1 \oplus B_2,$$

we call each $f_{ji} \colon A_i \to B_j$ a *component* of f, we call $\left(\begin{smallmatrix} f_{1i} \\ f_{2i} \end{smallmatrix}\right) \colon A_i \to B_1 \oplus B_2$ a *column* of f, and we call $\left(f_{j1}\ f_{j2}\right) : A_1 \oplus A_2 \to B_j$ a *row* of f. We use analogous terminology for larger matrices.

Remark 2.8. In a dagger finite biproduct category, an arrow of the form

$$\begin{array}{cc} & \begin{matrix} A_1 \oplus & \cdots & \oplus A_m \end{matrix} \\ B & \begin{pmatrix} f_1 & \cdots & f_m \end{pmatrix} \end{array}$$

is an isometry if and only if $f_i^\dagger \circ f_i = 1_{A_i}$ for all i and $f_j^\dagger \circ f_i = 0$ for $i \neq j$.

Remark 2.9. In a dagger additive category, every isometry is a component of a unitary. Indeed, suppose $f \colon A \to B$ is an isometry. Then the following arrow is unitary.

$$\begin{array}{cc} & \begin{matrix} A & \oplus & B \end{matrix} \\ \begin{matrix} A \\ \oplus \\ B \end{matrix} & \begin{pmatrix} 0 & f^\dagger \\ f & 1_B - f \circ f^\dagger \end{pmatrix} \end{array}$$

2.3 Dagger idempotents

Definition 2.10 (Dagger idempotents)**.** A arrow $p \colon A \to A$ is called a *dagger idempotent* (or *projection*) if $p = p \circ p = p^\dagger$.

Whenever $f \colon B \to A$ is an isometry, then $p = f \circ f^\dagger$ is a dagger idempotent. If p is of this form, we say that p is *dagger split*. When dagger splittings exist, they are unique up to unitary isomorphism. It is well-known that every dagger category can be fully embedded in a dagger category with all dagger splittings, called its *dagger idempotent completion* [24]. Moreover, all structure of interest (e.g., monoidal structure, biproducts, addition, negatives, and, as we will later introduce, pseudoinverses [5]) on a category transports to its dagger idempotent completion.

Definition 2.11 (Complementary idempotents)**.** Two idempotents $p, q \colon A \to A$ in a finite biproduct category are *complementary* if $p + q = 1_A$ and $q \circ p = 0 = p \circ q$.

If the category has negatives, complements always exist, because whenever p is a (dagger) idempotent, so is $1 - p$. It is obvious that the complement is unique in that case. Interestingly, uniqueness even holds without assuming negatives, because if both q_1 and q_2 are complements of p, we have $q_1 = (p + q_2) \circ q_1 = q_2 \circ q_1 = q_2 \circ (p + q_1) = q_2$.

Complementary dagger idempotents are an algebraic abstraction of orthogonal complement subspace projections.

Lemma 2.12 (Direct sum decomposition)**.** *Consider a (dagger) finite biproduct category in which all (dagger) idempotents split. Given an object A with complementary idempotents $p,q\colon A \to A$, there exist objects A_1, A_2 with $A = A_1 \oplus A_2$ such that*

$$p = \begin{array}{c} \scriptstyle A_1 \\ \scriptstyle \oplus \\ \scriptstyle A_2 \end{array} \overset{\scriptstyle A_1 \,\oplus\, A_2}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \quad \textit{and} \quad q = \begin{array}{c} \scriptstyle A_1 \\ \scriptstyle \oplus \\ \scriptstyle A_2 \end{array} \overset{\scriptstyle A_1 \,\oplus\, A_2}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}.$$

Moreover, the factorization is unique up to (unitary) isomorphisms of the direct sum factors.

Proof idea. Let A_1 be a splitting of p, and let A_2 be a splitting of q. The claimed properties are easy to verify. □

Remark 2.13. In Lemma 2.12 and elsewhere, we write $A = A_1 \oplus A_2$ instead of $A \cong A_1 \oplus A_2$; this is justified because (dagger) biproducts are defined up to (unitary) isomorphism in the first place.

2.4 Trace

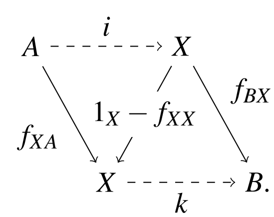
A *trace* on a symmetric monoidal category $\mathbf{C}$ is a family of operations $\mathrm{Tr}^X\colon \mathbf{C}(A\oplus X, B\oplus X) \to \mathbf{C}(A,B)$, subject to a small number of axioms that can be found in [10, 25]. The concept of a *partial trace* is defined similarly, except that Tr^X is a partially defined operation [6]. The axioms are such that a partially traced category in which the trace happens to be totally defined is a (totally) traced category. It was shown in [13, 14] that every partially traced category can be faithfully embedded in a totally traced one, and conversely, every monoidal subcategory of a totally traced category is partially traced.

We will make use of the following construction, which can be found in [14]. It is remarkable because it works in any additive category.

Definition 2.14 (Kernel-image trace)**.** Let $f\colon A\oplus X \to B\oplus X$ be an arrow in an additive category. The *kernel-image trace* $\mathrm{Tr}^X_{\mathrm{ki}} f\colon A \to B$ is defined if there exist arrows $i\colon A \to X$ and $k\colon X \to B$ such that

$$f_{XA} = (1_X - f_{XX})\circ i \qquad \text{and} \qquad k\circ(1_X - f_{XX}) = f_{BX},$$

as in the following commutative diagram



In this case, we define

$$\mathrm{Tr}^X_{\mathrm{ki}} f = f_{BA} + k\circ(1_X - f_{XX})\circ i.$$

(Otherwise, the kernel-image trace is undefined.) Note $\mathrm{Tr}^X_{\mathrm{ki}}$ is independent of the choice of each i and k, since

$$f_{BA} + f_{BX}\circ i = \mathrm{Tr}^X_{\mathrm{ki}} f = f_{BA} + k\circ f_{XA}.$$

Proposition 2.15 ([14])**.** *The kernel-image trace is a partial trace.*

Remark 2.16. In a dagger category, a (partial) trace is called a (partial) *dagger trace* if $\mathrm{Tr}(f^\dagger) = (\mathrm{Tr} f)^\dagger$. In a dagger additive category, the kernel-image trace is always a dagger partial trace, because its definition is self-dual.

3 Contractions

3.1 Basic properties

In the category of Hilbert spaces, a *contraction* is a map $f\colon A \to B$ such that for all $v \in A$, $\|f(v)\| \le \|v\|$. The following definition generalizes this concept to arbitrary dagger additive categories.

Definition 3.1 (Contraction)**.** A *contraction* in a dagger additive category is an arrow $f\colon A \to B$ such that $f^\dagger \circ f \le 1_A$. In other words, such that there exists an arrow $g\colon A \to B'$ with $f^\dagger \circ f + g^\dagger \circ g = 1_A$. Note that this is the case if and only if the map $\binom{f}{g}\colon A \to B \oplus B'$ is an isometry. A *cocontraction* is defined dually.

In particular, every isometry, coisometry, and unitary map is a contraction. Also, biproduct projections and injections are contractions.

Note that Definition 3.1 could be stated even without assuming negatives, but most of the useful properties of contractions rely on additivity. A point in case is the next proposition, which gives several alternative characterizations of contractions, none of which would be equivalent in the absence of negatives (see counterexamples D.14 and D.15).

Proposition 3.2 (Characterizations of contractions)**.** *Let $f\colon A \to B$ be an arrow in a dagger additive category. The following are equivalent.*

1. *f is a component of a unitary.*
2. *f is a contraction.*
3. *f is a cocontraction.*
4. *f is of the form $e \circ m$ for some isometry $m\colon A \to X$ and coisometry $e\colon X \to B$.*
5. *f is a composition of isometries and coisometries.*

We delay the proof until we have established some lemmas. The following lemma tells us contractions, like isometries, form a monoidal subcategory.

Lemma 3.3. *Contractions are closed under composition and monoidal products.*

Proof. For composition, let $f\colon A \to B$ and $g\colon B \to C$ be contractions. Then $f^\dagger \circ f \le 1_A$ and $g^\dagger \circ g \le 1_B$. Using Lemma 2.7(a), we get

$$(g \circ f)^\dagger \circ (g \circ f) = f^\dagger \circ g^\dagger \circ g \circ f \le f^\dagger \circ 1_B \circ f = f^\dagger \circ f \le 1_A.$$

Therefore, $f \circ g$ is a contraction. For monoidal products, let $f\colon A \to B$ and $g\colon A' \to B'$ be contractions. Using Lemma 2.7(b), we get

$$(f \oplus g)^\dagger \circ (f \oplus g) = (f^\dagger \circ f) \oplus (g^\dagger \circ g) \le 1_A \oplus 1_{A'} = 1_{A \oplus A'}.$$

Therefore, $f \oplus g$ is a contraction. □

Lemma 3.4 (Contractions as components of unitaries)**.** *In a dagger additive category, contractions are precisely the components of unitaries. In particular, contractions coincide with cocontractions.*

Proof. First, a component of a unitary is a composition of three contractions $u_{jk} = \pi_j \circ u \circ \iota_k$, and is therefore a contraction itself. Conversely, every contraction is a component of an isometry (as remarked in Definition 3.1), which in turn is a component of a unitary by Remark 2.9. Finally, since being a component of a unitary is a self-dual concept, so is being a contraction. □

We can now prove Proposition 3.2.

Proof of Proposition 3.2. The equivalence (1) $\iff$ (2) $\iff$ (3) is Lemma 3.4. For (2) $\implies$ (4), assume $f^\dagger \circ f + g^\dagger \circ g = 1$. Then $f = e \circ m$, where $e = \begin{pmatrix} 1 & 0 \end{pmatrix}$ is a coisometry and $m = \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)$ is an isometry. The implication (4) $\implies$ (5) is trivial, and (5) $\implies$ (2) follows because contractions are closed under composition by Lemma 3.3. □

3.2 Contractions and definiteness

Contractions have even better properties when the underlying dagger category satisfies the following condition.

Definition 3.5 (Definite)**.** A dagger category with a zero object is *definite* if for all arrows f, we have that $f^\dagger \circ f = 0$ implies $f = 0$.

In the familiar context of Hilbert spaces, the columns or rows of a contraction have norm at most 1. An analogue of this principle holds in any definite dagger additive category.

Lemma 3.6 (Maxed-out column)**.** *In a definite dagger additive category, assume $f = \left(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right)$ is a contraction. If f_1 is an isometry, then $f_2 = 0$.*

Proof. It suffices to show the result when f is an isometry, because every contraction is a row of an isometry. We have $1_A = f^\dagger \circ f = f_1^\dagger \circ f_1 + f_2^\dagger \circ f_2 = 1_A + f_2^\dagger \circ f_2$. Subtracting 1_A from both sides, we get $0 = f_2^\dagger \circ f_2$. Now by definiteness, $f_2 = 0$. □

Corollary 3.7 (Maxed-out row and column)**.** *In a definite dagger additive category, assume*

$$f = \begin{pmatrix} 1 & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

is a contraction. Then $f_{12} = 0$ and $f_{21} = 0$.

Proof. $f_{12} = 0$ follows from Lemma 3.6 and $f_{21} = 0$ follows from its dual. □

The first part of the following is basically Corollary 3.7 in more algebraic language. The second part amounts to the observation that the fixed points of a contraction f are also fixed by $f^\dagger$.

Corollary 3.8 (Fixed points of contraction)**.** *Suppose $f\colon A \to A$ is a contraction and $p\colon A \to A$ is a dagger idempotent in a definite dagger additive category.*

(a) If $p \circ f \circ p = p$, then $f \circ p = p = p \circ f$.

(b) $f \circ p = p$ if and only if $p \circ f = p$.

Proof. Without loss of generality, we can assume all dagger idempotents split, because otherwise we can pass to the dagger idempotent completion. Let $A = A_1 \oplus A_2$ be the decomposition of A obtained by splitting p and its complement as in Lemma 2.12. Write

$$f \;=\; \begin{array}{c} \\ A_1 \\ \oplus \\ A_2 \end{array} \!\!\begin{array}{c} A_1 \;\oplus\; A_2 \\ \begin{pmatrix} f_{11} & f_{12} \\ & \\ f_{21} & f_{22} \end{pmatrix} \end{array}.$$

To prove (a), note that $p \circ f \circ p = p$ means that $f_{11} = 1$, which by Corollary 3.7 implies that $f_{12} = 0$ and $f_{21} = 0$, hence $f \circ p = p = p \circ f$. Claim (b) follows from (a). □

4 Pseudoinverses

4.1 Definition of pseudoinverse

Every linear map $f\colon V \to W$ between finite-dimensional Hilbert spaces is of the form

$$f \;=\; \begin{array}{c} \\ \operatorname{im} f \\ \oplus \\ (\operatorname{im} f)^{\perp} \end{array} \overset{\textstyle (\ker f)^{\perp} \;\oplus\; \ker f}{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}},$$

where a is invertible. This section is about dagger additive categories in which an analogous fact holds. Observe that, given the above decomposition of $f\colon V \to W$, we automatically get a map $f^{\circ}\colon W \to V$ in the other direction via

$$f^{\circ} \;=\; \begin{array}{c} \\ (\ker f)^{\perp} \\ \oplus \\ \ker f \end{array} \overset{\textstyle \operatorname{im} f \;\oplus\; (\operatorname{im} f)^{\perp}}{\begin{pmatrix} a^{-1} & 0 \\ 0 & 0 \end{pmatrix}}.$$

We note that this "almost inverse" f° of f satisfies the following four properties:

$$f = f \circ f^{\circ} \circ f, \qquad f^{\circ} = f^{\circ} \circ f \circ f^{\circ}, \qquad f^{\circ} \circ f = (f^{\circ} \circ f)^{\dagger}, \qquad f \circ f^{\circ} = (f \circ f^{\circ})^{\dagger}. \tag{1}$$

It so happens that these four laws uniquely determine f° given f.

Definition 4.1 (Pseudoinverse)**.** In a dagger category, a *pseudoinverse* (or *Moore-Penrose pseudoinverse*) of a map $f\colon A \to B$ is an arrow $f^{\circ}\colon B \to A$ such that the equations (1) hold. A *pseudoinverse dagger category* (in [5], *Moore-Penrose dagger category*) is a dagger category in which every arrow has a pseudoinverse.

Before we prove uniqueness, here is a bit of background on pseudoinverses. They were introduced by Moore in [15] and rediscovered by Penrose in [17]. For an overview, see [4] or [2]. Pseudoinverses were studied in abstract dagger categories by Puystjens and Robinson in [18, 19, 20, 21, 22] and recently by Cockett and Lemay in [5].

Example 4.2. In **Hilb**, an arrow $f\colon \mathscr{H} \to \mathscr{H}'$ is pseudoinvertible if and only if the image of f is closed. In **FdHilb**, every arrow is pseudoinvertible.

We note the following equivalent characterization of pseudoinverses; it will simplify the proof of uniqueness in Proposition 4.4 below.

Lemma 4.3 (Second definition of pseudoinverse)**.** *Pseudoinverses f and f° in a dagger category are equivalently characterized by the equations*

$$f = f^{\circ\dagger} \circ f^{\dagger} \circ f, \qquad f = f \circ f^{\dagger} \circ f^{\circ\dagger}, \qquad f^{\circ} = f^{\dagger} \circ f^{\circ\dagger} \circ f^{\circ}, \qquad f^{\circ} = f^{\circ} \circ f^{\circ\dagger} \circ f^{\dagger}. \tag{2}$$

Proof. From (2), we derive

$$f \circ f^{\circ} = f^{\circ\dagger} \circ f^{\dagger} \circ f \circ f^{\circ} = f^{\circ\dagger} \circ f^{\dagger} \quad \text{and} \quad f^{\circ} \circ f = f^{\dagger} \circ f^{\circ\dagger} \circ f^{\circ} \circ f = f^{\dagger} \circ f^{\circ\dagger},$$

i.e., $f \circ f^{\circ} = (f \circ f^{\circ})^{\dagger}$ and $f^{\circ} \circ f = (f^{\circ} \circ f)^{\dagger}$. Hence the two definitions are equivalent as f and f° are permitted to slide past each other, picking up daggers. □

Proposition 4.4 (Uniqueness of pseudoinverse)**.** *If f° and $f^{\bullet}$ are both pseudoinverses of f, then $f^{\circ} = f^{\bullet}$.*

Proof. $f^\circ = f^\dagger \circ f^{\circ\dagger} \circ f^\circ = f^\bullet \circ f \circ f^\dagger \circ f^{\circ\dagger} \circ f^\circ = f^\bullet \circ f \circ f^\circ$. Symmetrically, $f^\bullet = f^\bullet \circ f \circ f^\circ$. □

Note that the notion of pseudoinverse is self-dual and therefore respected by dagger: if $f\colon A \to B$ is pseudoinvertible, then so is $f^\dagger$ with $(f^\dagger)^\circ = (f^\circ)^\dagger$. Also note that if f is pseudoinvertible, then $f \circ f^\circ$ and $f^\circ \circ f$ are dagger idempotents. More specifically, $f \circ f^\circ$ represents projection onto the image of f, and $f^\circ \circ f$ represents projection onto the coimage of f (i.e., the orthogonal complement of the kernel). We hence obtain the following decomposition, which is analogous to what happens in **FdHilb**.

Proposition 4.5 (Generalized singular value decomposition [5]). *Let $f\colon A \to B$ be an arrow in a dagger additive category in which all dagger idempotents split. Then f is pseudoinvertible if and only if we can write $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ such that*

$$f = \begin{array}{c} \scriptstyle B_1 \\ \scriptstyle\oplus \\ \scriptstyle B_2 \end{array} \overset{\scriptstyle A_1 \oplus A_2}{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}} \qquad \text{and} \qquad f^\circ = \begin{array}{c} \scriptstyle A_1 \\ \scriptstyle\oplus \\ \scriptstyle A_2 \end{array} \overset{\scriptstyle B_1 \oplus B_2}{\begin{pmatrix} a^{-1} & 0 \\ 0 & 0 \end{pmatrix}},$$

where $a\colon A_1 \to B_1$ is invertible. Moreover, the factorization of f is unique up to unitary isomorphisms of the direct sum factors.

Proof. Clearly, if f can be written in the stated form, then f is pseudoinvertible with pseudoinverse as stated. For the left-to-right implication, assume f is pseudoinvertible. Consider the dagger idempotents $f^\circ \circ f\colon A \to A$ and $f \circ f^\circ\colon B \to B$. By splitting them and their complements as in Lemma 2.12, we can write $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$, where $f^\circ \circ f = \iota_1^A \circ \pi_1^A$ and $f \circ f^\circ = \iota_1^B \circ \pi_1^B$. Let $a = \pi_1^B \circ f \circ \iota_1^A\colon A_1 \to B_1$. Then $f = f \circ f^\circ \circ f \circ f^\circ \circ f = \iota_1^B \circ \pi_1^B \circ f \circ \iota_1^A \circ \pi_1^A = \iota_1^B \circ a \circ \pi_1^A$, hence f is of the claimed form. Moreover, it is easy to verify that $a^{-1} = \pi_1^A \circ f^\circ \circ \iota_1^B$. Uniqueness is as in Lemma 2.12. □

4.2 EP maps

The generalized singular value decomposition of Proposition 4.5 is especially nice if f is a so-called EP-map, which we now define. This definition captures the notion of an endomorphism whose kernel and image are orthogonal complements.

Definition 4.6 (EP maps). An *EP map* (or *range hermitian map*) in a dagger category is a pseudoinvertible endomorphism $f\colon A \to A$ such that $f^\circ \circ f = f \circ f^\circ$.

Remark 4.7 (Normal operators are EP). If f is pseudoinvertible and $f^\dagger \circ f = f \circ f^\dagger$, then f is EP:

$$f^\circ \circ f = f^\dagger \circ f^{\circ\dagger} = f^\dagger \circ f \circ f^\circ \circ f^{\circ\dagger} = f^\dagger \circ f \circ (f^\dagger \circ f)^\circ = f \circ f^\dagger \circ (f \circ f^\dagger)^\circ = f \circ f^\dagger \circ f^{\circ\dagger} \circ f^\circ = f \circ f^\circ.$$

The following proposition characterizes EP maps in the style of Proposition 4.5.

Proposition 4.8. *Let $f\colon A \to A$ be an arrow in a dagger additive category in which all dagger idempotents split. Then f is EP if and only if we can write $A = A_1 \oplus A_2$ such that*

$$f = \begin{array}{c} \scriptstyle A_1 \\ \scriptstyle\oplus \\ \scriptstyle A_2 \end{array} \overset{\scriptstyle A_1 \oplus A_2}{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}},$$

where $a\colon A_1 \to A_1$ is invertible.

Proof. Like the proof of Proposition 4.5, but using the fact that the idempotents $f \circ f^\circ$ and $f^\circ \circ f$ are equal and therefore have the same splitting. □

Before we say more about EP maps, we need the following lemma.

Lemma 4.9. *In a dagger category with a zero object, if $f\colon A\to B$ is pseudoinvertible and $f^\dagger\circ f=0$, then $f=0$. In particular, every pseudoinverse dagger category with a zero object is definite.*

Proof. Using (2) from Lemma 4.3, we have $f=f^{\circ\dagger}\circ f^\dagger\circ f=0$. □

We saw in Corollary 3.8 that the fixed points of a contraction f are also fixed by $f^\dagger$. The following lemma is the same fact in different language: $g=1-f$ being EP means that $1-g^\circ\circ g$ (the projection onto the fixed points of f) is equal to $1-g\circ g^\circ$ (the projection onto the fixed points of $f^\dagger$).

Lemma 4.10 (Contractions and EP maps)**.** *Let $f\colon A\to A$ be a contraction in a pseudoinverse dagger additive category. Then $g=1_A-f$ is EP.*

Proof. Observe that $(1_A-g)\circ(1_A-g^\circ\circ g)=1_A-g^\circ\circ g$. By Lemma 4.9, the category is definite, and thus we can apply Corollary 3.8 to obtain $(1_A-g^\circ\circ g)\circ(1_A-g)=1_A-g^\circ\circ g$. Simplifying, we get $g^\circ\circ g\circ g=g$. Similarly, $g\circ g\circ g^\circ=g$, hence $g^\circ\circ g=g^\circ\circ g\circ g\circ g^\circ=g\circ g^\circ$, as claimed. □

5 Proof of the main result

The purpose of this section is to prove Theorem 1. That is, in a pseudoinverse dagger additive category, the monoidal subcategories of unitaries, isometries, coisometries, and contractions are traced. The proof in the case of isometries proceeds in two steps: In Lemma 5.1, we show that the kernel-image trace of a contraction (and therefore, of an isometry) is always defined. This is the only part of the proof that uses pseudoinverses. In Lemma 5.2, we show that the kernel-image trace of an isometry is again an isometry. These two facts imply that the kernel-image trace is totally defined on the category of isometries. Since it is already known to be a partial trace, these facts are sufficient to prove that the category of isometries is totally traced. The case of contractions is proved similarly, and the other cases are easy consequences.

Lemma 5.1 (Trace is defined for contractions)**.** *In a pseudoinverse dagger additive category, the kernel-image trace is always defined for contractions.*

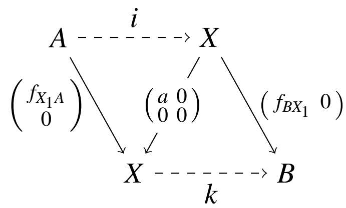
Proof. Let $f\colon A\oplus X\to B\oplus X$ be a contraction. Then f_{XX} is also a contraction, because it can be written as a composition of three contractions $f_{XX}=X\xrightarrow{\iota_X}A\oplus X\xrightarrow{f}B\oplus X\xrightarrow{\pi_X}X$. Then by Lemma 4.10, 1_X-f_{XX} is an EP map. For the rest of this proof, we assume that all idempotents split; this is without loss of generality because we can pass to the dagger idempotent completion. Note that the dagger idempotent completion still has pseudoinverses [5]. Because 1_X-f_{XX} is an EP map, by Proposition 4.8, we can write

$$1_X-f_{XX}=\begin{array}{c}\\ X_1\\ \oplus\\ X_2\end{array}\!\!\begin{array}{c}X_1\oplus X_2\\ \begin{pmatrix}a & 0\\ 0 & 0\end{pmatrix}\end{array},$$

where we have decomposed X into a sum of two objects $X_1\oplus X_2$ and $a\colon X_1\to X_1$ is invertible. Writing f in matrix form, we now have

$$f=\begin{array}{c}\\ B\\ \oplus\\ X_1\\ \oplus\\ X_2\end{array}\!\!\begin{array}{c}A\quad\oplus\quad X_1\quad\oplus\quad X_2\\ \begin{pmatrix}f_{BA} & f_{BX_1} & f_{BX_2}\\ f_{X_1A} & 1_{X_1}-a & 0\\ f_{X_2A} & 0 & 1_{X_2}\end{pmatrix}\end{array}.$$

Using Corollary 3.7, we get $f_{X_2A} = 0$ and $f_{BX_2} = 0$. To show that the kernel-image trace of f is defined, we must show that there exist i and k to complete the following diagram:



But this can be achieved with $i = \begin{pmatrix} a^{-1} \circ f_{X_1A} \\ 0 \end{pmatrix}$ and $k = \begin{pmatrix} f_{BX_1} \circ a^{-1} & 0 \end{pmatrix}$. □

In the next lemma, we do not assume pseudoinverses, so the kernel-image trace of a given isometry may not exist. However, we show that if it does exist, it is an isometry.

Lemma 5.2 (Trace of isometry)**.** *In a dagger additive category, the kernel-image trace of an isometry, if it exists, is an isometry.*

Proof. Consider an arrow $f\colon A \oplus X \to B \oplus X$ with components

$$f = \begin{array}{c} \\ B \\ \oplus \\ X \end{array} \begin{array}{c} A \quad \oplus \quad X \\ \begin{pmatrix} f_{BA} & f_{BX} \\ & \\ f_{XA} & f_{XX} \end{pmatrix} \end{array}$$

Assume that f is an isometry, so that we have

$$\begin{aligned} f_{BX}^\dagger \circ f_{BX} + f_{XX}^\dagger \circ f_{XX} &= 1_X, \\ f_{BA}^\dagger \circ f_{BA} + f_{XA}^\dagger \circ f_{XA} &= 1_A, \\ f_{BX}^\dagger \circ f_{BA} + f_{XX}^\dagger \circ f_{XA} &= 0. \end{aligned} \tag{3}$$

Also assume that $\mathrm{Tr}_{\mathrm{ki}}^X f$ exists, so in particular there exists $i\colon A \to X$ satisfying

$$f_{XA} = (1_X - f_{XX}) \circ i. \tag{4}$$

To show that $\mathrm{Tr}_{\mathrm{ki}}^X f$ is an isometry, we calculate

$$\begin{aligned}
(\mathrm{Tr}_{\mathrm{ki}}^X f)^\dagger \circ \mathrm{Tr}_{\mathrm{ki}}^X f &= (f_{BA}^\dagger + i^\dagger \circ f_{BX}^\dagger) \circ (f_{BA} + f_{BX} \circ i) \\
&= f_{BA}^\dagger \circ f_{BA} + f_{BA}^\dagger \circ f_{BX} \circ i + i^\dagger \circ f_{BX}^\dagger \circ f_{BA} + i^\dagger \circ f_{BX}^\dagger \circ f_{BX} \circ i \\
\text{(by (3))} \quad &= f_{BA}^\dagger \circ f_{BA} - f_{XA}^\dagger \circ f_{XX} \circ i - i^\dagger \circ f_{XX}^\dagger \circ f_{XA} + i^\dagger \circ (1_X - f_{XX}^\dagger \circ f_{XX}) \circ i \\
\text{(by (4))} \quad &= f_{BA}^\dagger \circ f_{BA} - i^\dagger \circ (1_X - f_{XX}^\dagger) \circ f_{XX} \circ i - i^\dagger \circ f_{XX}^\dagger \circ (1_X - f_{XX}) \circ i + i^\dagger \circ (1_X - f_{XX}^\dagger \circ f_{XX}) \circ i \\
&= f_{BA}^\dagger \circ f_{BA} + i^\dagger \circ (-f_{XX} + f_{XX}^\dagger \circ f_{XX}) \circ i + i^\dagger \circ (-f_{XX}^\dagger + f_{XX}^\dagger \circ f_{XX}) \circ i + i^\dagger \circ (1_X - f_{XX}^\dagger \circ f_{XX}) \circ i \\
&= f_{BA}^\dagger \circ f_{BA} + i^\dagger \circ (1_X - f_{XX}^\dagger - f_{XX} + f_{XX}^\dagger \circ f_{XX}) \circ i \\
&= f_{BA}^\dagger \circ f_{BA} + i^\dagger \circ (1_X - f_{XX}^\dagger) \circ (1_X - f_{XX}) \circ i \\
\text{(by (4))} \quad &= f_{BA}^\dagger \circ f_{BA} + f_{XA}^\dagger \circ f_{XA} \\
\text{(by (3))} \quad &= 1_A.
\end{aligned}$$
□

Note that the proof only used the i of the kernel-image trace and not the k. We use this fact to immediately obtain the following.

Lemma 5.3 (Trace of contraction)**.** *In a dagger additive category, the kernel-image trace of a contraction, if it exists, is a contraction.*

Proof. Suppose $f\colon A\oplus X\to B\oplus X$ is a contraction. By the definition of contraction, there exists an object B' and an arrow $g\colon A\oplus X\to B'$ such that $f^\dagger\circ f+g^\dagger\circ g=1_{A\oplus X}$, or in other words, such that

$$h=\begin{array}{c} \\ B' \\ \oplus \\ B \\ \oplus \\ X \end{array}\overset{\begin{array}{ccc}A & \oplus & X\end{array}}{\begin{pmatrix} g_{B'A} & g_{B'X} \\ f_{BA} & f_{BX} \\ f_{XA} & f_{XX} \end{pmatrix}}$$

is an isometry. Now assume that the kernel-image trace of f exists. While this does not necessarily imply that the kernel-image trace of h exists, we nevertheless get the existence of $i\colon A\to X$ such that $f_{XA}=(1_X-f_{XX})\circ i$. As seen in the proof of Lemma 5.2, this is sufficient to show that

$$\begin{pmatrix} g_{B'A} \\ f_{BA} \end{pmatrix}+\begin{pmatrix} g_{B'X} \\ f_{BX} \end{pmatrix}\circ i=\begin{pmatrix} g_{B'A}+g_{B'X}\circ i \\ f_{BA}+f_{BX}\circ i \end{pmatrix}$$

is an isometry. Thus, the kernel-image trace of f, which is $f_{BA}+f_{BX}\circ i$, is a contraction, as claimed. □

We are now ready to prove our main theorem.

Proof of Theorem 1. Lemmas 5.1, 5.2, and 5.3 show that the kernel-image trace of the ambient category is total on isometries and contractions. Dually, the same holds for coisometries. Hence the trace is also total on their intersection, the unitaries. Moreover, in the cases of unitaries and contractions, the trace is a dagger trace by Remark 2.16. □

Conclusion and future work

We showed that in every pseudoinverse dagger additive category, each of the subcategories of isometries, coisometries, unitary maps, and contractions forms a totally traced category. This generalizes a result by Bartha in the case of finite dimensional Hilbert spaces. One of the main ingredients of this construction is the notion of pseudoinverse, which was originally studied for matrices by Moore and Penrose, but makes sense in any dagger category. Contractions can also be defined in any dagger category (as compositions of isometries and coisometries), but they only behave as expected if one assumes additive structure and definiteness. The latter follows from the existence of pseudoinverses.

One might ask whether Bartha's trace has a physical interpretation. We do not know the answer, but some potential evidence to the contrary is that the trace on contractions is not a continuous operation, and that it does not exist in infinite dimensional spaces. See Appendix B for more details.

In future work, it would be interesting to investigate whether the assumptions under which contractions are traced could be further reduced. In fact, there are examples of dagger additive categories in which the contractions are totally traced but not all pseudoinverses exist; see Counterexample D.1. On the other hand, it is not sufficient to merely assume, say, the existence of dagger kernels; see Remark B.2.

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A The pseudotrace is not a trace

In the proof of our main theorem, pseudoinverses play a minor, but crucial role: they are only used to prove that the kernel-image trace is total. In Bartha's original work [3], pseudoinverses play a larger part, because he uses them directly to define the trace on the category of finite dimensional Hilbert spaces and isometries via the following formula:

$$\mathrm{Tr}^X_{\mathrm{ps}} f = f_{BA} + f_{BX} \circ (1_X - f_{XX})^{\circ} \circ f_{XA}. \tag{5}$$

In fact, the above formula is defined for all linear maps f (not necessarily isometries), and, as we show below, it agrees with the kernel-image trace whenever the latter exists. However, Bartha's operation is not a trace on the category of all linear maps, because it fails to satisfy the trace axioms. We call it the *pseudotrace*.

Definition A.1 (Pseudotrace)**.** In a dagger additive category, the *pseudotrace* of $f\colon A \oplus X \to B \oplus X$ is defined by (5), if the pseudoinverse of $1_X - f_{XX}$ exists, and undefined otherwise. In particular, if the category has pseudoinverses, this is a totally defined operation.

Warning A.2. In general, the pseudotrace is not a (partial or total) trace. In **FdHilb**, let $X = \mathbb{C}^2$ and $A = \mathbb{C}$, and consider $f\colon A \oplus X \to A \oplus X$ and $g\colon X \to X$ defined by

$$f = \left(\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) \qquad \text{and} \qquad g = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Then we find

$$\mathrm{Tr}^X_{\mathrm{ps}}((1_A \oplus g) \circ f) = 0 \qquad \text{and} \qquad \mathrm{Tr}^X_{\mathrm{ps}}(f \circ (1_A \oplus g)) = -1,$$

violating dinaturality (one of the laws of traces) [14].

Lemma A.3 (Pseudotrace and kernel-image trace)**.** *In a dagger additive category, whenever the pseudotrace and kernel-image trace are both defined, they coincide.*

Proof. Let $f\colon A \oplus X \to B \oplus X$ be an arrow with both $\mathrm{Tr}^X_{\mathrm{ki}} f$ and $\mathrm{Tr}^X_{\mathrm{ps}} f$ defined. Taking i and k as in Definition 2.14, we have

$$\begin{aligned} \mathrm{Tr}^X_{\mathrm{ki}} f &= f_{BA} + k \circ (1_X - f_{XX}) \circ i \\ &= f_{BA} + k \circ (1_X - f_{XX}) \circ (1_X - f_{XX})^{\circ} \circ (1_X - f_{XX}) \circ i \\ &= f_{BA} + f_{BX} \circ (1_X - f_{XX})^{\circ} \circ f_{XA} \\ &= \mathrm{Tr}^X_{\mathrm{ps}} f. \end{aligned}$$

□

B Non-physicality of the trace?

One may ask whether the trace operation on Hilbert spaces and unitaries (or isometries, or contractions) has a physical interpretation, e.g., whether there is some physical device that can perform this operation when presented with an input unitary in the form of a black box. One potential issue is that the trace is not a continuous operation, i.e., an infinitesimal variation in the input may cause a large variation in the output. Another potential issue is that neither the pseudotrace (Definition A.1) nor the kernel-image trace is total on infinite dimensional Hilbert spaces, even when restricted to contractions, isometries, coisometries, or unitaries. The next two remarks make this precise.

Remark B.1 (Non-continuity of trace)**.** The trace on finite-dimensional Hilbert spaces with unitary maps is not a continuous operation. Take for example the θ-parameterized family of rotations

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

The trace along the second row and column is 1 for $\theta = 0$ but is -1 for $0 < \theta < 2\pi$. On the other hand, the trace is continuous on strict contractions, because in that case the pseudoinverse in (5) is an actual inverse, which is a continuous operation.

Remark B.2 (Nonexistence of trace in infinite dimensions)**.** Let $f\colon \ell^2 \to \ell^2$ be the contraction on the Hilbert space of square-summable sequences that multiplies the nth term of every sequence by $\frac{1}{n}$. Consider the unitary map $\ell^2 \oplus \ell^2 \to \ell^2 \oplus \ell^2$ defined by

$$\begin{pmatrix} -(1_{\ell^2} - f) & \sqrt{1_{\ell^2} - (1_{\ell^2} - f)^2} \\ \sqrt{1_{\ell^2} - (1_{\ell^2} - f)^2} & 1_{\ell^2} - f \end{pmatrix}.$$

The pseudotrace along the second row and column does not exist, as f is not pseudoinvertible (i.e., f does not have a closed image; see Example 4.2). Indeed, if there were an induced invertible map from the coimage of f (here the entire space) to the image of f, then its inverse would have to be unbounded. Neither does the kernel-image trace exist, as $\sqrt{1_{\ell^2} - (1_{\ell^2} - f)^2}$ does not factor through f. Indeed, if there were k with $\sqrt{1_{\ell^2} - (1_{\ell^2} - f)^2} = k \circ f$, then k would have to be unbounded.

C More on pseudoinverses

Pseudoinverses arise inevitably in relation to dagger idempotents. Recall that a morphism of idempotents $f\colon p \to q$ is an arrow f such that $q \circ f \circ p = f$. As shown in [5], the pseudoinvertible arrows in any dagger category **C** exactly correspond to isomorphisms of dagger idempotents (that is, isomorphisms in the dagger idempotent completion of **C**):

Proposition C.1 (Pseudoinverses via dagger idempotents [5])**.** *In a dagger category, an arrow $f\colon A \to B$ is pseudoinvertible if and only if there are dagger idempotents $p\colon A \to A$ and $q\colon B \to B$ such that f is an isomorphism of dagger idempotents $f\colon p \to q$. Furthermore, the inverse isomorphism of dagger idempotents $q \to p$ is given by f°, and we have $p = f^\circ \circ f$ and $q = f \circ f^\circ$.*

Proof. To say $f\colon p \to q$ is an isomorphism of dagger idempotents with inverse $g\colon q \to p$ means $f = q \circ f \circ p$ and $g = p \circ g \circ q$ with $g \circ f = p$ and $f \circ g = q$. Since p and q are dagger idempotents, we have $(g \circ f)^\dagger = g \circ f$ and $(f \circ g)^\dagger = f \circ g$. Moreover $f \circ g \circ f = f$ and $g \circ f \circ g = g$, so $g = f^\circ$.

Conversely, assume f is pseudoinvertible and let $p = f^\circ \circ f$ and $q = f \circ f^\circ$. We have that $f\colon p \to q$ and $f^\circ\colon q \to p$ are morphisms of dagger idempotents, because $f = f \circ f^\circ \circ f \circ f^\circ \circ f$ and $f^\circ = f^\circ \circ f \circ f^\circ \circ f \circ f^\circ$. The compositions $f^\circ \circ f$ and $f \circ f^\circ$ are respectively the identities on p and q. □

Unlike inverses, pseudoinverses do not in general compose; see Counterexample D.5. Still, we do have the following.

Corollary C.2 (Composition of pseudoinverses)**.** *In a dagger category, if $f\colon A \to B$ and $g\colon B \to C$ are pseudoinvertible with $f \circ f^\circ = g^\circ \circ g$, then $g \circ f$ is pseudoinvertible with $(g \circ f)^\circ = f^\circ \circ g^\circ$. Moreover, $g \circ f \circ f^\circ \circ g^\circ = g \circ g^\circ$ and $f^\circ \circ g^\circ \circ g \circ f = f^\circ \circ f$.*

Proof. This follows from Proposition C.1, as a composition of isomorphisms of idempotents is an isomorphism of idempotents. □

In a dagger additive category, we obtain four dagger idempotents of interest (written in the matrix form of Proposition 4.5, assuming that the dagger splittings exist):

$$f\circ f^\circ = \begin{array}{c} \scriptstyle \mathrm{im} f \oplus (\mathrm{im} f)^\perp \\ \scriptstyle \mathrm{im} f \oplus (\mathrm{im} f)^\perp \end{array} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad f^\circ\circ f = \begin{array}{c} \scriptstyle (\ker f)^\perp \oplus \ker f \\ \scriptstyle (\ker f)^\perp \oplus \ker f \end{array} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$1 - f\circ f^\circ = \begin{array}{c} \scriptstyle \mathrm{im} f \oplus (\mathrm{im} f)^\perp \\ \scriptstyle \mathrm{im} f \oplus (\mathrm{im} f)^\perp \end{array} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad 1 - f^\circ\circ f = \begin{array}{c} \scriptstyle (\ker f)^\perp \oplus \ker f \\ \scriptstyle (\ker f)^\perp \oplus \ker f \end{array} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

They are, respectively, the projections onto the image, the coimage, the cokernel, and the kernel of f. The following propositions show that these names are justified.

Proposition C.3 (Image via pseudoinverse)**.** *Let $f\colon A \to B$ be a pseudoinvertible arrow in a dagger category such that $f\circ f^\circ$ splits via the mono $m\colon X \to B$. Then m is the image of f (i.e., the universal subobject through which f factors).*

Proof. We have $m = f\circ f^\circ \circ m$, so m factors through every arrow that f factors through. □

Note that f° and $f^\dagger$ have the same image projection, namely $f^\circ \circ f = f^\dagger \circ f^{\dagger\circ}$. This is also the coimage projection of f (and of $f^{\dagger\circ}$). Dually, f° and $f^\dagger$ have the same coimage projection, namely $f\circ f^\circ = f^{\dagger\circ}\circ f^\dagger$, which is also the image projection of f (and of $f^{\dagger\circ}$).

Proposition C.4 (Kernel via pseudoinverse)**.** *Let $f\colon A \to B$ be a pseudoinvertible arrow in a dagger finite biproduct category, with p and $f^\circ\circ f$ complementary dagger idempotents. Then f has a (dagger) kernel (in the standard sense of [7]) if and only if p (dagger) splits. The splitting is given by the inclusion map of the kernel.*

Proof. Observe that for all arrows $m\colon X \to A$, we have $f\circ m = 0$ if and only if $m = p\circ m$. Indeed, if $f\circ m = 0$, then $m = (f^\circ \circ f + p)\circ m = p\circ m$. Conversely, if $m = p\circ m$ then $f\circ m = f\circ f^\circ\circ f\circ p\circ m = 0$. A universal such arrow m is equivalently characterized as a kernel of f or as an equalizer of p and 1_A. But an equalizer of an idempotent and an identity is the same as a mono splitting the idempotent, as desired. Moreover, a dagger kernel is by definition a kernel that is an isometry. It suffices to observe that every splitting of a dagger idempotent by an isometry m is a dagger splitting:

$$e = e\circ m^\dagger \circ m = m^\dagger \circ e^\dagger \circ m^\dagger = m^\dagger.$$ □

Lemma C.5 (Pseudoinvertible monos split)**.** *In a dagger category, every pseudoinvertible mono (dually, epi) is split by its pseudoinverse.*

Proof. Suppose $f\colon A \to B$ is a pseudoinvertible mono. We have $f = f\circ f^\circ \circ f$. Thus by cancellation $f^\circ\circ f = 1_A$. □

Proposition C.6. *Every pseudoinverse dagger additive category in which all dagger idempotents split is an abelian category (in the standard sense of [12]).*

Proof. By Proposition C.4, all kernels exist. Moreover, every mono m is normal as m is a kernel of $1 - m \circ m^{\circ}$, using both Proposition C.4 and Lemma C.5. Dually, all cokernels exist and every epi is normal. □

Although this paper is about pseudoinverse dagger additive categories, so far we have not given many examples.

Example C.7. Let $\mathbb{F}$ be any dagger subfield of $\mathbb{C}$ (i.e., a subfield closed under conjugation). Then $\mathbf{Mat}(\mathbb{F})$ is a pseudoinverse dagger additive category. Indeed, it was observed in [16] that the pseudoinverse of a matrix f over an arbitrary dagger field exists if and only if $\mathrm{rank}(f) = \mathrm{rank}(f^{\dagger} \circ f) = \mathrm{rank}(f \circ f^{\dagger})$, and the rank of a matrix does not change upon passing to a larger field.

Lemma C.8. *In a pseudoinverse dagger additive category, $\mathbb{Q}$ embeds into the endomorphism ring at each non-zero object.*

Proof. Let $n \in \mathbb{N}_{\geq 1}$, and let $\delta\colon A \to A^n$ be the canonical n-ary diagonal map. By symmetry, we have $\delta^{\circ} = \begin{pmatrix} d & \cdots & d \end{pmatrix}$ for some $d\colon A \to A$. Since δ is mono, by Lemma C.5, we have $\delta^{\circ} \circ \delta = 1_A$, hence $n \cdot d = 1_A$. Thus $n \cdot 1_A$ has a multiplicative inverse. As $\mathbb{Q}$ is the universal ring in which every natural number has a multiplicative inverse, and $\mathbb{Q}$ has no proper quotients, the result follows. □

Example C.9 (Free pseudoinverse dagger additive categories)**.** Since pseudoinverse dagger additive categories are essentially algebraic, we can consider freely generated structures. It follows from Lemma C.8 that the free pseudoinverse dagger additive category on an object is equivalent to $\mathbf{Mat}(\mathbb{Q})$. But more exotic examples exist, such as the free pseudoinverse dagger category on an object with an endomorphism; see Counterexample D.2.

D Counterexamples

This section consists of several counterexamples, which preclude various strengthenings of results in the paper. Our main theorem gives a sufficient condition for the monoidal subcategory of contractions in a dagger additive category to be traced: the existence of pseudoinverses. However, this is not a necessary condition.

Counterexample D.1 (Trace without pseudoinverses)**.** The kernel-image trace on the dagger additive category $\mathbf{Mat}(\mathbb{Z})$ of integer-valued matrices is totally defined on contractions. Indeed, since contractions are equivalently submatrices of unitaries, they are the matrices with entries in $\{-1,0,1\}$ with at most one nonzero entry per row and per column, and one may check that all kernel-image traces of such matrices exist. However, not all arrows (even those of the form $1 - f$ with f a contraction) are pseudoinvertible, e.g., the matrix $\begin{pmatrix} 2 \end{pmatrix}$.

Given the examples we have seen so far, it is reasonable to ask whether all pseudoinverse dagger additive categories are sub dagger additive categories of matrices over the complex numbers. However, this is not the case.

Counterexample D.2 (Non complex matrix pseudoinverse dagger additive category)**.** The free pseudoinverse dagger additive category on an object $*$ and an arrow $f\colon * \to *$ does not embed into any dagger additive category of matrices over a field. Indeed, given an endomorphism f of a finite dimensional vector space, the image eventually stabilizes with repeated application, i.e. (assuming relevant pseudoinverses exist), $f^n \circ (f^n)^{\circ} = f^{n+1} \circ (f^{n+1})^{\circ}$ for some n. But such n can be arbitrarily high, so in the free instance this cannot happen for any particular n.

We saw in Remark 2.9 that in a dagger additive category, every isometry is a component of a unitary. If moreover all dagger idempotents dagger split, then every isometry is a *column* of a unitary, using Lemma 2.12. We note that this stronger statement does not hold without the assumption of dagger splittings.

Counterexample D.3 (Non unitary-column isometry)**.** Consider the full dagger finite biproduct subcategory of **FdHilb** of spaces with dimension not equal to 1. The inclusion of a 2-dimensional subspace into a 3-dimensional space is then not a column of a unitary.

We saw in Corollary 3.7 that for a contraction in a definite dagger additive category, any row or column with a 1 has all other entries 0. This does not hold without assuming definiteness.

Counterexample D.4 (Non maxed-out row)**.** In $\mathbf{Mat}(\mathbb{F}_2)$, the matrix $\begin{pmatrix}1 & 1 & 1\end{pmatrix}$ is a coisometry.

Next, we give some more counterexamples relating to pseudoinverses. Pseudoinverses do not compose: in general we do not have $(g\circ f)^\circ = f^\circ \circ g^\circ$. However, this does hold when the image projection of f and the coimage projection of g coincide, as in Corollary C.2. We observe that it is not sufficient to merely assume that the image projection of f factors through the coimage projection of g.

Counterexample D.5 (Non composition of pseudoinverses)**.** Consider the dagger idempotent $p = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$ and the invertible matrix $a = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ in **FdHilb**. We have $a\circ p = p$, and thus $(a\circ p)^\circ = p^\circ = p$, whereas $p^\circ \circ a^\circ = p\circ a^{-1} = \left(\begin{smallmatrix}1 & -1\\ 0 & 0\end{smallmatrix}\right)$.

On the other hand, the next example shows that it is still possible to have $(g\circ f)^\circ = f^\circ \circ g^\circ$ in cases where the image projection of f and coimage projection of g are incompatible (do not even commute); in particular, the sufficient condition given in Corollary C.2 is not necessary. (See [5] for a necessary and sufficient condition for pseudoinverses to compose.)

Counterexample D.6 (Composition of pseudoinverses)**.** In the dagger category of finite sets and relations (i.e., boolean-valued matrices), the pseudoinverse of the idempotent $p = \left(\begin{smallmatrix}1 & 0\\ 1 & 0\end{smallmatrix}\right)$ is the idempotent $p^\circ = \left(\begin{smallmatrix}1 & 1\\ 0 & 0\end{smallmatrix}\right)$. Hence $(pp)^\circ = p^\circ \circ p^\circ$. However, the image projection $p\circ p^\circ = \left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)$ does not commute with the coimage projection $p^\circ \circ p = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$.

We saw in Lemma C.5 that every pseudoinvertible mono is a split mono. However, the converse does not hold in general.

Counterexample D.7 (Non pseudoinvertible split mono)**.** In $\mathbf{Mat}(\mathbb{Z})$, $m = \left(\begin{smallmatrix}1\\ 1\end{smallmatrix}\right)$ is a split mono. If its pseudoinverse existed, it would also be the pseudoinverse in $\mathbf{Mat}(\mathbb{Q})$. However, the pseudoinverse in $\mathbf{Mat}(\mathbb{Q})$ is $\left(\frac{1}{2}\ \frac{1}{2}\right)$, which is not in $\mathbf{Mat}(\mathbb{Z})$.

The following counterexamples show that when we do not assume the existence of all pseudoinverses, the pseudotrace (Definition A.1) may be defined in cases where the kernel-image trace is undefined or vice versa (even restricting to unitaries).

Counterexample D.8 (Non pseudotrace kernel-image trace)**.** In $\mathbf{Mat}(\mathbb{Z})$, the unitary $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ does not have a pseudotrace along the second row and column, as $1-(-1) = 2$ is not pseudoinvertible. It does have a kernel-image trace equal to $1+0(2)0 = 1$.

Counterexample D.9 (Non kernel-image trace pseudotrace)**.** In $\mathbf{Mat}(\mathbb{Z}[x]/\langle x^2\rangle)$, the unitary $\left(\begin{smallmatrix}-1 & x\\ x & 1\end{smallmatrix}\right)$ has pseudotrace along the second row and column, as $1-1=0$ is pseudoinvertible. It does not have a kernel-image trace, as x is not of the form $0\circ i$ for any i.

Finally, we give several counterexamples that show certain results about dagger additive categories do not hold in the absence of negatives. It follows from Proposition C.4 that any isometry m in a dagger additive category is the dagger kernel of $1-m\circ m^\dagger$. However, isometries need not be kernels if we do not assume the existence of negatives.

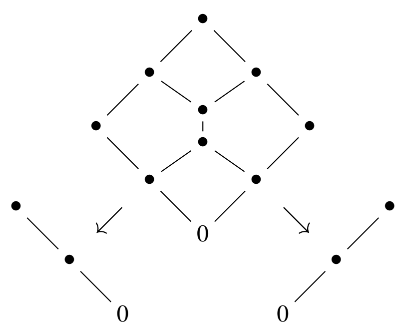
Counterexample D.10 (Non kernel isometry)**.** In the dagger finite biproduct category of finite sets and relations (i.e., boolean-valued matrices), the isometry $\begin{pmatrix}1\\1\end{pmatrix}$ is not a kernel.

We saw in Lemma 2.12 that complementary split idempotents are tantamount to direct sum decompositions. In an additive category, idempotents p and q are complementary if and only if $p+q=1$, which does not hold in the absence of negatives.

Counterexample D.11 (Non direct sum idempotent sum)**.** Consider the (dagger) finite biproduct category of finite sets and relations. Letting $p=q=1_{\{*\}}$, we have $p+q=1_{\{*\}}$, but $pq\neq 0$. Thus, p and q are not complementary. Also, both p and q are split via $\{*\}$, but $\{*\}$ is not isomorphic to $\{*\}\oplus\{*\}$.

Complementary split idempotents are split by (co)kernels of one another, by an argument similar to the proof of Proposition C.4. Thus each idempotent can be recovered from the other as the (necessarily unique) idempotent split by its kernel and cokernel. Hence, in the absence of negatives, it is reasonable to ask whether idempotents that are split by (co)kernels of one another are necessarily complementary. However, this is not the case.

Counterexample D.12 (Non complementary mutual (co)kernels)**.** In the finite biproduct category of bounded semilattices (equivalently, modules over the booleans), consider the following semilattice, with evident "projection-like" idempotents onto the shown sublattices:

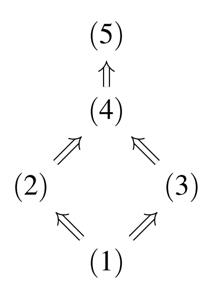


These idempotents are split by (co)kernels of one another, but they are not complementary.

The kernel-image trace is a partial trace on any additive category. It is reasonable to ask whether the same formula works in an arbitrary finite biproduct category, simply leaving the trace undefined where the relevant subtraction is not defined. However, this does not give a partial trace in general.

Counterexample D.13 (Kernel-image non trace)**.** Consider $\mathbf{Mat}(\mathbb{N}[x,y]/\langle xy\rangle)$. The matrix $\begin{pmatrix}xy\end{pmatrix}=0$ has a negative, so the kernel-image trace formula (tracing out the entire input and output) $0+0(1-0)0$ is defined. On the other hand, the matrix $\begin{pmatrix}yx\end{pmatrix}$ does not have a negative, so the kernel-image trace formula is undefined. This violates the dinaturality law for partial traces [14].

In Proposition 3.2 we saw five equivalent characterizations of contractions in a dagger additive category. However, these conditions are not equivalent in a mere dagger finite biproduct category (i.e., without assuming negatives). In fact, they are all distinct, with implications between them as follows:



To distinguish them, it suffices to see that (2) is distinct from (3) and that (4) is distinct from (5); it is then clear that the self-dual definitions are distinct from the non-self-dual definitions. To say (3) is distinct from (4) means that contractions are not the same as cocontractions.

Counterexample D.14 (Non cocontraction contraction)**.** Consider the dagger finite biproduct category of finite sets and relations (i.e., boolean-valued matrices). The isometries are the matrices featuring at least one 1 per column and at most one 1 per row. The matrix $\binom{1}{1}$ is an isometry, thus a contraction, but not a component of a coisometry, thus not a cocontraction.

To say (4) is distinct from (5) means that not every coisometry followed by an isometry is equal to an isometry followed by a coisometry.

Counterexample D.15 (Non isometry then coisometry coisometry then isometry)**.** Consider $\mathbb{N}[x,x^\dagger]\langle x^\dagger x = 1\rangle$ (the free dagger rig on an isometry x). Its elements have explicit normal forms, as finite expressions $\sum_{i,j\geq 0} n_{i,j} x^j (x^\dagger)^i$. In the corresponding dagger finite biproduct category of matrices, the isometries are the matrices having entries in $\{0,1,x,x^2,\ldots\}$ with one nonzero entry per column and at most one nonzero entry per row. Clearly the matrix $\begin{pmatrix} xx^\dagger \end{pmatrix}$ cannot be expressed as an isometry followed by a coisometry.