Line Graphs and Circulants

Jason Brown and Richard Hoshino
Department of Mathematics and Statistics
Dalhousie University
Halifax, Nova Scotia, Canada B3H 3J5

Abstract

The line graph of G, denoted $L(G)$, is the graph with vertex set $E(G)$, where vertices x and y are adjacent in $L(G)$ iff edges x and y share a common vertex in G. In this paper, we determine all graphs G for which $L(G)$ is a circulant graph. We will prove that if $L(G)$ is a circulant, then G must be one of three graphs: the complete graph K_4, the cycle C_n, or the complete bipartite graph $K_{a,b}$, for some a and b with $\gcd(a,b) = 1$.

Keywords: circulant graph, line graph

1 Introduction

The *line graph* of a simple graph G, denoted $L(G)$, is the graph with vertex set $E(G)$, where vertices x and y are adjacent in $L(G)$ iff edges x and y share a common vertex in G.

Line graphs make important connections between many important areas of graph theory. For example, determining a maximum matching in a graph is equivalent to finding a maximum independent set in the corresponding line graph. Similarly, edge colouring is equivalent to vertex colouring in the line graph.

Much research has been done on the study and application of line graphs; a comprehensive survey of results is found in [6].

Whitney [7] solved the determination problem for line graphs, by showing that with the exception of the graphs $K_{1,3}$ and K_3, a graph is uniquely characterized by its line graph.

Theorem 1.1 ([7]) *Let G and H be two connected graphs for which $L(G) \simeq L(H)$. If $\{G, H\} \neq \{K_{1,3}, K_3\}$, then $G \simeq H$.*

By Whitney's Theorem, we will refer to G as *the* corresponding graph of $L(G)$, whenever $L(G) \not\simeq K_3$.

Let Φ be any mapping from the set of finite graphs to itself. For example, the line graph operator L is such a mapping. A natural question is to determine all families of graphs Γ for which Γ is closed under Φ.

This question was investigated in [6] for $\Phi = L$, where the author surveyed known families of graphs Γ for which $G \in \Gamma$ implies $L(G) \in \Gamma$. As a simple example, the family of regular graphs is closed under L, since the line graph of an r-regular graph G is a $(2r-2)$-regular graph. Other L-closed families include k-connected graphs, non-chordal graphs, non-perfect graphs, non-comparability graphs [1], and Eulerian graphs [3].

Given the context of the Prisner survey, we sought to investigate whether the class of circulant graphs is also closed under the line graph operator. This question provided the motivation for the results of this paper.

Circulants are highly symmetric graphs, and are a subset of the more general family of Cayley graphs. Each circulant graph is characterized by its order n, and its *generating set* $S \subseteq \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. We define $G = C_{n,S}$ to be the graph with vertex set $V(G) = \mathbb{Z}_n$ and edge set $E(G) = \{uv : |u-v|_n \in S\}$, where $|x|_n = \min\{|x|, n-|x|\}$ is the *circular distance* modulo n. By definition, $|x|_n \equiv \pm x \pmod{n}$. Furthermore, note that if $n = ab$, then $|at|_{ab} = a|t|_b$ for all t.

The following two lemmas are well-known and easy to prove.

Lemma 1.2 ([2]) *Let $C_{n,S}$ have generating set $S = \{s_1, s_2, \ldots, s_m\}$. Then $C_{n,S}$ is connected iff $d = \gcd(n, s_1, s_2, \ldots, s_m) = 1$.*

Lemma 1.3 ([5]) *Consider the circulant graphs $C_{n,S}$ and $C_{n,T}$. If $T = rS = \{|rs|_n : s \in S\}$ for some integer r, then $C_{n,S} \simeq C_{n,T}$.*

As an example, we can readily verify that if G is the complete graph K_4, then $L(K_4) \simeq C_{6,\{1,2\}}$.

Due to the regularity and symmetry of a circulant graph, a natural conjecture is that $L(G)$ is a circulant whenever G is a circulant. As we see above, such is the case for $G = K_4 = C_{4,\{1,2\}}$. It is also true for the n-cycle $C_n = C_{n,\{1\}}$. However, a counterexample to the conjecture is found for $G = K_5$.

Theorem 1.4 *$L(K_5)$ is not a circulant graph.*

<u>Proof</u> Since K_5 has 10 edges, $L(K_5)$ has 10 vertices. Suppose on the contrary that $L(K_5)$ is a circulant. Then, $L(K_5) = C_{10,S}$ for some generating set $S \subseteq \{1, 2, 3, 4, 5\}$. Since K_5 is 4-regular, this implies that $L(K_5)$

is 6-regular, and hence $|S| = 3$. Furthermore, $5 \notin S$, as otherwise $L(K_5)$ would have odd degree. Thus, S must be one of $\{1,2,3\}$, $\{1,3,4\}$, $\{1,2,4\}$, or $\{2,3,4\}$. We may reject the latter two cases since $C_{10,S}$ has a 5-clique (namely the set $\{0,2,4,6,8\}$), while $L(K_5)$ clearly has no 5-clique.

Therefore, $S = \{1,2,3\}$ or $S = \{1,3,4\}$. By Lemma 1.3, $C_{10,\{1,2,3\}} \simeq C_{10,\{1,3,4\}}$ with the multiplier $r = 3$, which implies that $L(K_5)$ must be isomorphic to $C_{10,\{1,2,3\}}$. This circulant contains ten distinct 4-cliques, namely the cliques $\{i, i+1, i+2, i+3\}$ for $0 \leq i \leq 9$, where each element is reduced mod 10. This implies that $L(K_5)$ must also have ten 4-cliques. However, $L(K_5)$ has only 5 cliques of cardinality 4, since a 4-clique must arise from four pairwise adjacent edges in K_5, and this occurs iff all four edges share a common vertex in K_5. Therefore, no such generating set S exists. $\square$

We have given examples of graphs G for which $L(G)$ is a circulant, but shown that $G = K_5$ does not satisfy this property. Is it possible to characterize all graphs G such that $L(G)$ is a circulant? We answer this question fully in Section 2.

Before we proceed with the main theorem, let us describe one more family of graphs for which its line graph is a circulant. Recall that $K_{a,b}$ is the complete bipartite graph, which has bipartition (X, Y) with $|X| = a$ and $|Y| = b$.

Theorem 1.5 *Let $G = K_{a,b}$, where $\gcd(a,b) = 1$. Then, $L(G) \simeq C_{ab,S}$, where $S = \{1 \leq k \leq \lfloor \frac{ab}{2} \rfloor : a|k \text{ or } b|k\}$.*

Proof Let (X, Y) be the bipartition of G, with $|X| = a$ and $|Y| = b$. Represent each edge in G by an ordered pair (x, y), where $0 \leq x \leq a-1$ and $0 \leq y \leq b-1$. We will label each edge xy in G with the integer $e_{x,y} := bx + ay \pmod{ab}$. Thus, edge (x,y) in G will correspond to the *vertex* $e_{x,y}$ in $L(G)$.

We claim that $e_{x,y}$ is one-to-one. On the contrary, suppose that $e_{x,y} = e_{x',y'}$ for some $(x,y) \neq (x',y')$. Then $b(x-x') \equiv a(y'-y) \pmod{ab}$. Since $\gcd(a,b) = 1$, we must have $a|(x-x')$ and $b|(y'-y)$. But $0 \leq x, x' \leq a-1$ and $0 \leq y, y' \leq b-1$, and so this implies that $(x,y) = (x',y')$, a contradiction. Therefore, the vertices of $L(G)$ are the integers from 0 to $ab-1$, inclusive.

Vertices $e_{x,y}$ and $e_{x',y'}$ are adjacent in $L(G)$ iff $x = x'$ or $y = y'$. In the former case, we have $|e_{x,y} - e_{x',y'}|_{ab} = |ay - ay'|_{ab} = a|y-y'|_b$, and in the latter case, $|e_{x,y} - e_{x',y'}|_{ab} = |bx - bx'|_{ab} = b|x-x'|_a$. Hence, $e_{x,y} \sim e_{x',y'}$ in $L(G)$ iff $|e_{x,y} - e_{x',y'}|_{ab} \in S$, where S is the union of all possible values of $a|y-y'|_b$ and $b|x-x'|_a$. Note that $1 \leq |y-y'|_b \leq \lfloor \frac{b}{2} \rfloor$ and $1 \leq |x-x'|_a \leq \lfloor \frac{a}{2} \rfloor$. Then this implies that S takes on every multiple of a and b less than or equal to $\lfloor \frac{ab}{2} \rfloor$. Hence,

$$S = \left\{ 1 \leq k \leq \left\lfloor \frac{ab}{2} \right\rfloor : a|k \text{ or } b|k \right\}.$$

We conclude that $L(G)$ is isomorphic to the circulant $C_{ab,S}$. $\square$

As an example, if $G = K_{7,12}$, then $L(G) \simeq C_{84,\{7,12,14,21,24,28,35,36,42\}}$. We note that Theorem 1.5 fails when $\gcd(a, b) \neq 1$. As an example, consider $G = K_{3,3}$. Since G is 3-regular, $L(G)$ must be a 4-regular graph on 9 vertices. Suppose that $L(G)$ is a circulant. Then, we must have $L(G) \simeq C_{9,S}$ for some generating set $S \subseteq \{1, 2, 3, 4\}$ with $|S| = 2$. And a simple case analysis shows that no such S satisfies $C_{9,S} \simeq L(K_{3,3})$.

We have now shown that $L(G)$ is a (connected) circulant if $G = K_4$, $G = C_n$, or $G = K_{a,b}$ for some $\gcd(a, b) = 1$. What is surprising is that these are the *only* such possibilities. The rest of the paper is devoted to proving this result.

Theorem 1.6 *Let G be a connected graph such that $L(G)$ is a circulant. Then G must either be C_n, K_4, or $K_{a,b}$ for some a and b with* $\gcd(a, b) = 1$.

2 Proof of The Main Theorem

If G is connected, then so is $L(G)$. So let us assume that $L(G) = C_{n,S}$ is a connected circulant graph.

If i is a vertex of $L(G)$, then the corresponding edge in G will be denoted e_i. Thus, $x \sim y$ in $L(G)$ iff e_x and e_y share a common vertex in G.

First consider the case when 1 is an element of the generating set S.

Theorem 2.1 *If $L(G) = C_{n,S}$ and $1 \in S$, then G must be $K_{1,n}$, C_n, or K_4.*

Proof If $S = \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, then $L(G) = K_n$. This implies that $G = K_{1,n}$ for all n (and in the special case $n = 3$, we could also have $G = K_3 = C_3$).

So assume $L(G) \neq K_n$. Then, there must exist a *smallest* index k such that $1, 2, \ldots, k \in S$ and $k + 1 \notin S$. Note that $k \leq \lfloor \frac{n}{2} \rfloor - 1$. We split our analysis into three subcases.

Case 1: $3 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$.

The vertices $\{0, 1, 2, \ldots, k\}$ induce a copy of K_{k+1} in the line graph $L(G)$, since $1, 2, \ldots, k \in S$. Therefore, the edges in $\{e_0, e_1, e_2, \ldots, e_k\}$ must be pairwise adjacent in G. Since $k \geq 3$, these $k + 1$ edges must share a common vertex u in G. Now consider edge e_{k+1}. This edge is adjacent to e_i for each $1 \leq i \leq k$, and thus, shares a common vertex with each of these k edges. Since $k \geq 3$, u must also be an endpoint of e_{k+1}. But then $e_0 \sim e_{k+1}$, which contradicts the assumption that $k + 1 \notin S$. Thus, no graph G exists in this case.

<u>*Case 2:*</u> $k = 2$.

First note that if $n \leq 5$, then $L(G) = K_n$, so suppose that $n \geq 6$. If $n = 6$, then $L(G) = C_{6,\{1,2\}}$, from which we immediately derive $G = K_4$ (this result was also quoted in the introduction to this section). So assume that $n \geq 7$. Consider the subgraph of G induced by the edges $\{e_0, e_1, e_2, e_3, e_{n-3}, e_{n-2}, e_{n-1}\}$.

If $1, 2 \in S$ and $3 \notin S$, we claim that this subgraph of G must be isomorphic to one of the graphs in Figure 1. For notational convenience, we represent edge e_k by just the index k.

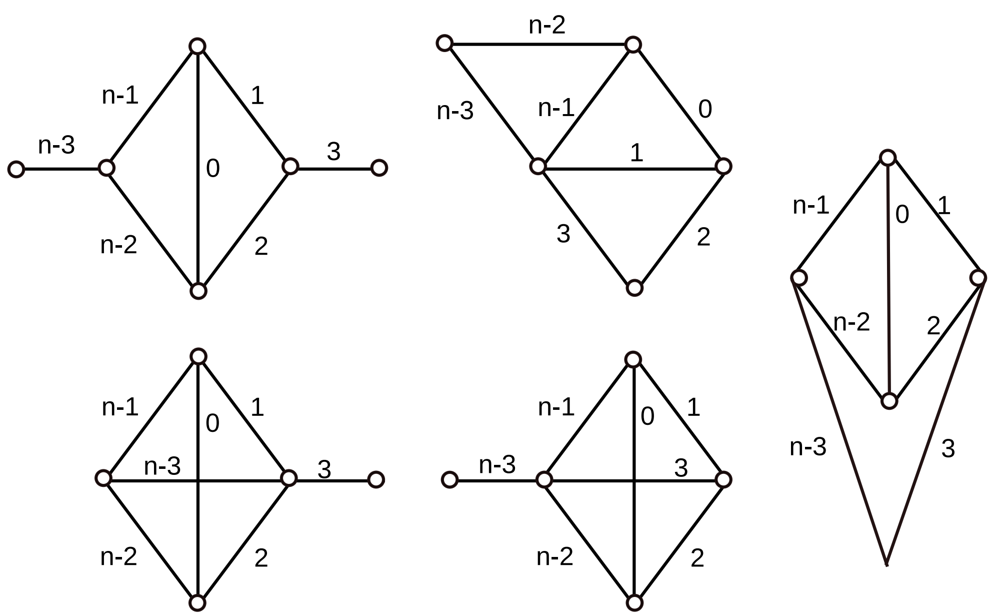

Figure 1: Possible subgraphs of G induced by these 7 edges.

To explain why this subgraph of G must be isomorphic to one of these five graphs, we perform a step-by-step case analysis. Start with the edges e_0, e_1, and e_2. Either these three edges induce a K_3 or a $K_{1,3}$. In each case, add edge e_3. Since $3 \notin S$, e_3 is adjacent to e_1 and e_2, but not e_0. Now add e_{n-1}. This edge is adjacent to e_0 and e_1, but not e_2. At this stage, we have three possible cases, as illustrated in Figure 2.

Now add edge e_{n-2}, which is adjacent to e_0 and e_{n-1}, but not e_1. Finally, add edge e_{n-3}, which is adjacent to e_{n-2} and e_{n-1}, but not e_0. Adding these two edges in all possible ways to our three graphs in Figure 2, we find that there are five possible subgraphs. These five subgraphs correspond to the

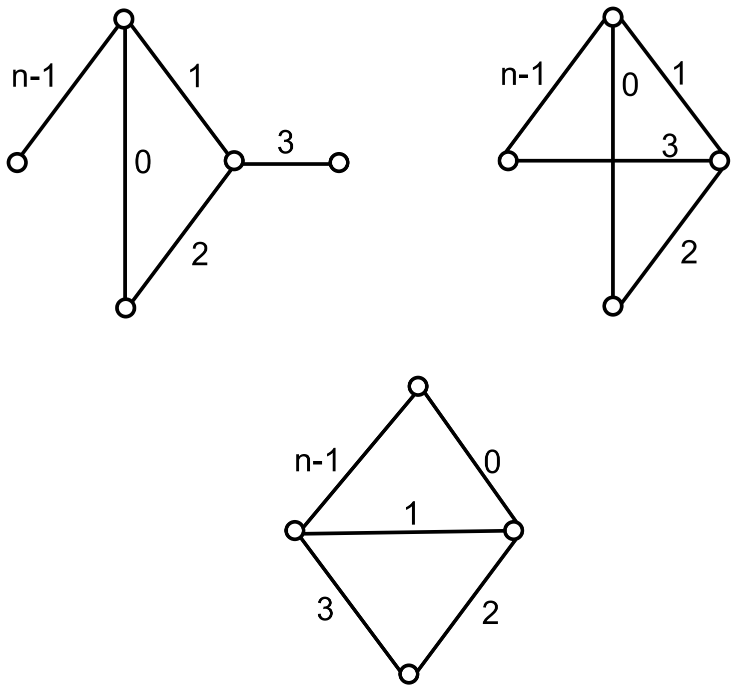

Figure 2: Possible subgraphs of G induced by $\{e_0, e_1, e_2, e_3, e_{n-1}\}$.

graphs in Figure 1.

If $n = 7$, then $e_1 \sim e_4$ in the second graph of Figure 1 (top centre) and $e_2 \sim e_5$ in other four. But this contradicts the assumption that $3 \notin S$. So assume $n \geq 8$. In the second graph, $e_{n-3} \sim e_1$ and $e_{n-2} \not\sim e_2$, which shows that $4 \in S$ and $4 \notin S$, a contradiction. We get a similar contradiction for the other four graphs: $e_{n-2} \sim e_2$ and either $e_{n-3} \not\sim e_1$ or $e_{n-1} \not\sim e_3$.

So in the case $k = 2$, we *must* have $n = 6$. Thus, $L(G) = C_{6,\{1,2\}}$, implying that $G = K_4$.

Case 3: $k = 1$.

If $S = \{1\}$, then $L(G) = C_n$, and so $G = C_n$ (in the special case that $n = 3$, we could also have $G = K_{1,n}$). So we may assume that $|S| > 1$ and that $n \geq 4$. We know that $2 \notin S$ since $k = 1$. Let l be the smallest index for which $1 \in S$, $2, 3, \ldots, l \notin S$, and $l + 1 \in S$. Note that $2 \leq l \leq \lfloor \frac{n}{2} \rfloor - 1$.

The vertices $\{0, 1, \ldots, l + 1\}$ induce a copy of C_{l+2} in the line graph $L(G)$, since $2, 3, \ldots, l \notin S$. Since $l \geq 2$, the edges $\{e_0, e_1, \ldots, e_{l+1}\}$ must induce an $(l+2)$-cycle in G. Let x be the vertex shared by e_0 and e_1, and let y be the vertex shared by e_{l+1} and e_0.

Now consider e_{l+2}. Since $e_{l+2} \sim e_{l+1}$ and $e_{l+2} \not\sim e_l$, one of the endpoints of e_{l+2} must be y. Since $e_{l+2} \sim e_1$ and $e_{l+2} \not\sim e_2$, one of the endpoints of e_{l+2} must be x. But then this forces $e_{l+2} = xy = e_0$, which is a contradiction, since $l+2 \leq \lfloor \frac{n}{2} \rfloor + 1 < n$. Thus, no graph G exists in this case. $\square$

By Theorem 2.1, we have proven that if $1 \in S$ and $L(G)$ is a circulant, then G must be $K_{1,n}$, K_4 or C_n. Now consider all generating sets S with $1 \notin S$.

Suppose $L(G) = C_{n,S}$ with some element $x \in S$ such that $\gcd(x,n) = 1$. There must exist an integer y with $xy \equiv 1 \pmod{n}$. Then the set $yS = \{|yi|_n : i \in S\}$ is a generating set with $|S|$ elements, and by Lemma 1.3, $C_{n,S} \simeq C_{n,yS}$. Since $1 \in yS$, we have reduced the problem to the previously-solved case of $1 \in S$.

Therefore, we may assume that every $i \in S$ satisfies $\gcd(i,n) > 1$. We prove the following.

Theorem 2.2 *Let $L(G) = C_{n,S}$ be a circulant graph. If every $i \in S$ satisfies $\gcd(i,n) > 1$, then G must be the complete bipartite graph $K_{a,b}$, where a and b are integers for which $\gcd(a,b) = 1$ and $n = ab$.*

Then Theorem 1.6 follows an immediate corollary of Theorems 2.1 and 2.2.

The proof of Theorem 2.2 is quite technical as we require multiple lemmas, and a very careful treatment of the Extreme Principle. The rest of the paper is devoted to proving Theorem 2.2.

Let $S = \{s_1, s_2, \ldots, s_m\}$. By Lemma 1.2, $\gcd(n, s_1, s_2, \ldots, s_m) = 1$, or else $G = C_{n,S}$ is disconnected. For every integer t with $\gcd(t,n) = 1$, define

$$tS = \{|tx|_n \ : \ x \in S\} = \{t_1, t_2, \ldots, t_m\}.$$

We claim that there exists an integer $t \geq 1$ so that $\gcd(t_1, t_2, \ldots, t_m) = 1$.

If $\gcd(s_1, s_2, \ldots, s_m) = 1$, then this claim is trivial, since we can set $t = 1$. So suppose $\gcd(s_1, s_2, \ldots, s_m) = d > 1$. Note that $\gcd(n, s_1, s_2, \ldots, s_m) = \gcd(n,d) = 1$. Therefore, there must exist an integer $t \geq 1$ such that $td \equiv 1 \pmod{n}$. Then, $ts_i = td \cdot \frac{s_i}{d} \equiv \frac{s_i}{d} \pmod{n}$ for each $1 \leq i \leq m$, implying that $t_i = |ts_i|_n = \frac{s_i}{d}$. Hence, $\gcd(t_1, t_2, \ldots, t_m) = \frac{1}{d}\gcd(s_1, s_2, \ldots, s_m) = 1$.

Hence, we have proven the existence of such an index t. Therefore, by Lemma 1.3, $L(G) = C_{n,S} \simeq C_{n,tS}$, with $\gcd(t_1, t_2, \ldots, t_m) = 1$.

For each $2 \leq k \leq m$, consider all k-tuples $(a_1, a_2, \ldots, a_k)$ comprised of the elements of tS so that $\gcd(a_1, a_2, \ldots, a_k) = 1$. Clearly such a k-tuple exists for $k = m$ by setting $a_i = t_i$ for each $1 \leq i \leq k = m$.

Lemma 2.3 *Consider all k-tuples $(a_1, a_2, \ldots, a_k)$ for which each $a_i \in tS$ and $\gcd(a_1, a_2, \ldots, a_k) = 1$. Of all k-tuples that satisfy these conditions (over all $k \geq 2$), consider the k-tuple for which the sum $a_1 + a_2 + \ldots + a_k$ is minimized. This minimum k-tuple must satisfy $k = 2$.*

Proof Suppose on the contrary that the minimum k-tuple satisfies $k \geq 3$. Then $L(G) = C_{n,tS}$ is a connected circulant graph with vertex 0 adjacent to each of a_1, a_2, and a_3. Consider e_0 in the corresponding graph G. We know that e_{a_1}, e_{a_2}, and e_{a_3} share a common vertex with e_0. By the Pigeonhole Principle, two of these three edges must share the same common vertex, and hence $|a_j - a_i|_n \in tS$ for some $1 \leq i < j \leq 3$.

Without loss, say $a_2 - a_1 \in tS$. We have $\gcd(a_1, a_2) = \gcd(a_1, a_2 - a_1)$. If $a_2 - a_1 = a_j$ for some $1 \leq j \leq k$, then $\gcd(a_1, a_3, a_4, \ldots, a_k) = 1$, contradicting the minimality of our chosen k-tuple. If $a_2 - a_1$ does not already appear as some a_j in our minimum k-tuple, then $\gcd(a_1, a_2 - a_1, a_3, a_4, \ldots, a_k) = 1$, and once again we have contradicted our minimality assumption. $\square$

Lemma 2.3 shows that in a minimum k-tuple satisfying the given conditions, we must have $k = 2$. This minimum k-tuple must be a pair (a, b), where $a + b$ is minimized over all pairs such that $a, b \in tS$ and $\gcd(a, b) = 1$. Without loss, assume $a < b$. Specifically, this choice of (a, b) implies that $b - a \notin tS$, as otherwise the pair $(a, b - a)$ satisfies $\gcd(a, b - a) = 1$ and contradicts the minimality of (a, b). Since $1 \notin tS$, we have $2 \leq a < b \leq \lfloor \frac{n}{2} \rfloor$.

Lemma 2.4 *Let $L(G) = C_{n,tS}$ be a circulant. For this minimum pair (a, b) for which $a, b \in tS$ and $\gcd(a, b) = 1$, we have $|a + b|_n \notin tS$.*

On the contrary, suppose that $|a + b|_n \in tS$. Consider the subgraph of G induced by the edges in the set $\{e_0, e_a, e_b, e_{b-a}, e_{a+b}, e_{2a}\}$. Since $2a < a + b < n$ and $\gcd(a, b) = 1$, these six edges are distinct.

From $a, b, |a + b|_n \in tS$ and $b - a \notin tS$, a simple case analysis shows that this subgraph must be isomorphic to K_4, with one of two possible edge labellings, as shown in Figure 3. We arrive at this conclusion by considering the edges in the following order: e_0, e_a, e_b, e_{a+b}, e_{b-a}, and e_{2a}. After we have included five edges, there are four possible subgraphs. But after we include e_{2a}, we must eliminate the two cases with $e_{b-a} \not\sim e_a$, and this leaves us with the two labellings in Figure 3. As before, we represent edge e_k by just the index k for notational convenience.

In both valid labellings, $e_0 \sim e_{2a}$. Therefore, if $|a + b|_n \in tS$, this implies that $|2a|_n \in tS$ as well.

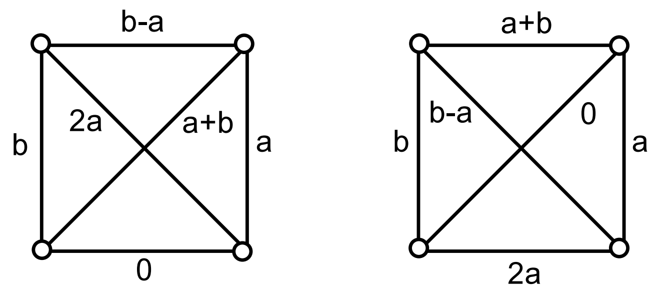

Figure 3: Two possible edge labellings of K_4.

Now consider the edge e_{n-a}. We claim that edge e_{n-a} is distinct from the other six edges. Note that $\gcd(a,n) > 1$, $\gcd(b,n) > 1$, and $\gcd(a,b) = 1$, with $2 \le a < b \le \frac{n}{2}$. If $n-a$ equals 0, a, b or $b-a$, then we have an immediate contradiction. If $n-a = a+b$, then $n = 2a+b$, so that $\gcd(a,n) = \gcd(a, 2a+b) = \gcd(a,b) = 1$, by the Euclidean algorithm. But then $\gcd(a,n) = 1$, which is a contradiction. Finally, if $n-a = 2a$ (i.e., $n = 3a$), we argue that (a,b) is not the *minimum* pair satisfying the given conditions. Let $b' = |a+b|_n \in tS$. Since $a < b$, we have $a+b > 2a > \frac{n}{2}$, and so $b' = |a+b|_n = n-(a+b) = 2a-b$. Then, $\gcd(a,b') = \gcd(a, 2a-b) = \gcd(a,b) = 1$. Note that $b' = 2a-b < b$. So (a,b') is a pair satisfying $\gcd(a,b') = 1$ and $a, b' \in tS$, thus contradicting the minimality of (a,b).

Thus, edge e_{n-a} is distinct from the six other edges in this K_4 subgraph, and is adjacent to each of e_0, e_a, and e_b. But the three edges $\{e_b, e_0, e_a\}$ induce the path P_4, and so e_{n-a} must coincide with one of the edges e_{a+b}, e_{b-a}, or e_{2a}. This establishes our desired contradiction, and so we have shown that $|a+b|_n \notin tS$. $\square$

We have now shown that $a, b \in tS$, $\gcd(a,b) = 1$, $b-a \notin tS$, and $|a+b|_n \notin tS$. We will now prove that in our circulant $L(G) = C_{n,tS}$, n must equal ab, and that the generating set tS must equal

$$tS = \left\{ 1 \le k \le \left\lfloor \frac{ab}{2} \right\rfloor : a|k \text{ or } b|k \right\}.$$

By Lemma 1.5 and Theorem 1.1, this will immediately establish Theorem 2.2. Hence, it suffices to prove that $n = ab$, and that $1 \le k \le \lfloor \frac{ab}{2} \rfloor$ is an element of tS iff k is a multiple of a or b.

Now consider the subgraph of $L(G)$ induced by the vertices $\{0, a, b, n-a, n-b\}$. It is well-known (and straightforward to show) that any line graph

$L(G)$ is *claw-free*, i.e., $L(G)$ has no induced $K_{1,3}$ subgraph. This implies that $|2a|_n \in tS$, as otherwise $\{0, a, b, n-a\}$ induces a $K_{1,3}$ subgraph in $L(G)$, since $b - a \notin tS$ and $|a+b|_n \notin tS$. Similarly, if $n > 2b$, then $|2b|_n \in tS$ as well. In the exceptional case that $n = 2b$ (i.e., $b = \frac{n}{2} = \lfloor \frac{n}{2} \rfloor$), we have $b = n - b$, and we will deal with this case separately.

We have shown that in our generating set tS, if $a \in tS$, then $|2a|_n \in tS$. We now prove that n must be a multiple of a.

Lemma 2.5 *Let $a \in tS$ be the element as described above. Then $a|n$.*

Proof Since $a < \lfloor \frac{n}{2} \rfloor$, we know that $n > 2a$. Consider two cases.

Case 1: $|3a|_n \in tS$.

We will show that n must be a multiple of a, and that $ka \in tS$ for each $1 \leq k \leq \lfloor \frac{n}{2a} \rfloor$. The subgraph of G induced by the edges $\{e_0, e_a, e_{2a}, e_{3a}\}$ must be isomorphic to $K_{1,4}$, since these edges are pairwise adjacent. Let u be the vertex common to each edge. Since e_{4a} is adjacent to e_a, e_{2a}, and e_{3a}, it follows that u must also be an endpoint of e_{4a}, which implies that $|4a|_n \in tS$ since $e_0 \sim e_{4a}$. Continuing in this manner, we see that each $|ka|_n \in tS$, for all $k \geq 4$. Now, let $d = \gcd(a, n)$. Then, there exists an integer m for which $|ma|_n = d$, which implies that $d \in tS$. If $d < a$, then (d, b) is a pair with $\gcd(d, b) = 1$ since $d = \gcd(a, n)|a$. And this contradicts the minimality of (a, b). Therefore, we must have $d = a$, which implies that $a|n$. Hence, $|ka|_n \in tS$ for each $k \geq 1$. In other words, $ka \in tS$ for each $1 \leq k \leq \lfloor \frac{n}{2a} \rfloor$.

Case 2: $|3a|_n \notin tS$.

We prove that if $|3a|_n \notin tS$, then $n = ka$ for some $3 \leq k \leq 6$. Consider the subgraph of G induced by the edges $\{e_0, e_a, e_{2a}, e_{3a}, e_{4a}, e_{5a}, e_{6a}\}$, where the indices are reduced mod n (if necessary). We now split our analysis into two subcases: when n does not divide ma for any $m \leq 6$, and when $n|ma$ for some $m \leq 6$.

If n does not divide ma for any $m \leq 6$, then these seven edges must be distinct. We claim that if $|3a|_n \notin tS$, then the edges $\{e_0, e_a, e_{2a}, e_{3a}, e_{4a}, e_{5a}\}$ must induce a copy of K_4, with one of two possible edge-labellings as shown in Figure 4. In both possible edge labellings, $e_0 \sim e_{4a}$, i.e., $|4a|_n \in tS$.

This is justified by doing a case analysis, considering the edges in the following order: e_0, e_a, e_{2a}, e_{3a}, e_{4a}, and e_{5a}. After the first five edges have been included, there are three possible subgraphs. But after we include e_{5a}, we see that we must eliminate the subgraph with $e_0 \not\sim e_{4a}$. This leaves us with the two subgraphs in Figure 4.

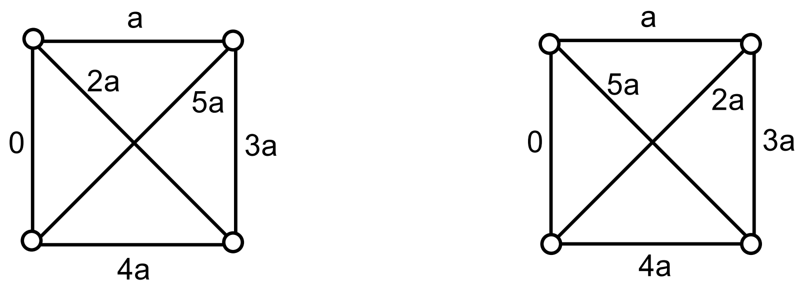

Figure 4: Two possible K_4 subgraphs induced by $\{e_0, e_a, e_{2a}, e_{3a}, e_{4a}, e_{5a}\}$.

Now consider e_{6a}. We know that $e_{6a} \not\sim e_{3a}$, while e_{6a} is adjacent to each of e_{2a}, e_{4a}, and e_{5a}. And this implies that e_{6a} and e_0 coincide, which is a contradiction.

Therefore, n must divide ma for some $m \leq 6$. If $n < 6a$, then this reduces to the previously solved Case 1, where we showed that $ka \in tS$ for $1 \leq k \leq \lfloor \frac{n}{2a} \rfloor$. Thus, Case 2 only adds one possible scenario not previously considered, namely the case $n = 6a$ and $3a \notin tS$.

In both cases, we have shown that n must be a multiple of a. $\square$

By Lemma 2.5, we have shown that $n \equiv 0 \pmod a$ and that $ka \in tS$ for each $1 \leq k \leq \lfloor \frac{n}{2a} \rfloor$, with the only possible exception being the case when $3a \notin tS$ and $n = 6a$. We have an analogous result when we replace a by b, except in the special case $n = 2b$. Thus, we have shown that $L(G) = C_{n,tS}$ *must* satisfy one of the following four cases.

1. $n = 6a$, with $a, 2a \in tS$, $3a \notin tS$, and $lb \in tS$ for $1 \leq l \leq \lfloor \frac{n}{2b} \rfloor$.
2. $n = 6b$, with $b, 2b \in tS$, $3b \notin tS$, and $ka \in tS$ for $1 \leq k \leq \lfloor \frac{n}{2a} \rfloor$.
3. $n = 2b$, with $b \in tS$, and $ka \in tS$ for $1 \leq k \leq \lfloor \frac{n}{2a} \rfloor$.
4. $n = mab$ for some integer m, with $ka \in tS$ for $1 \leq k \leq \lfloor \frac{n}{2a} \rfloor$ and $lb \in tS$ for $1 \leq l \leq \lfloor \frac{n}{2b} \rfloor$.

Note that the third case is a special instance of the fourth case (when $m = 1$ and $a = 2$), so we may disregard this case as we will include it in our analysis of the fourth case. We first prove that the first two cases cannot occur, leaving us with only Case 4 to consider. In this remaining final case, we will prove that n must equal ab and that

$$tS = \left\{1 \leq k \leq \left\lfloor \frac{ab}{2} \right\rfloor : a|k \text{ or } b|k\right\}.$$

As mentioned previously, this implies the conclusion of Theorem 2.2.

We show that the first two cases are impossible. By symmetry, we will just disprove the first case. As mentioned before, the subgraph of G induced by the edges $\{e_0, e_a, e_{2a}, e_{3a}, e_{4a}, e_{5a}\}$ must be isomorphic to K_4, since $a, 2a \in tS$ and $3a \notin tS$. There are two possible labellings of the edges on K_4, as shown in Figure 4. Now consider the edges e_b and e_{a+b}, which are distinct from the six edges of the subgraph since $\gcd(a, b) = 1$ and $a, b > 1$. Since $b - a \notin tS$, we must have $e_b \not\sim e_a$ and $e_{a+b} \not\sim e_{2a}$. Also, we must have $e_b \sim e_{a+b}$, $e_b \sim e_0$, and $e_{a+b} \sim e_a$. Therefore, the only possible edge labellings are given in Figure 5.

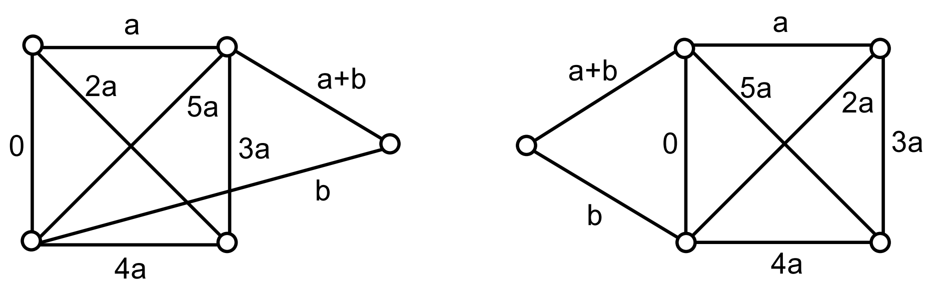

Figure 5: Two possible subgraphs induced by this set of eight edges.

The first graph has $e_{a+b} \sim e_{3a}$ and $e_b \not\sim e_{2a}$, and the second graph has $e_{a+b} \not\sim e_{3a}$ and $e_b \sim e_{2a}$. Therefore, in both graphs, we have $|2a - b|_n \sim tS$ and $|2a - b|_n \not\sim tS$, a contradiction. Thus, we have proven that the first two possible cases for $L(G) = C_{n,tS}$ are impossible, and so we only need to consider the fourth and final case.

We have $n = mab$ for some integer m, where $ka \in tS$ for $1 \leq k \leq \lfloor \frac{n}{2a} \rfloor$ and $lb \in tS$ for $1 \leq l \leq \lfloor \frac{n}{2b} \rfloor$. We now prove that $n = ab$, i.e., $m = 1$.

Suppose that $m > 1$. Since $b > a > 1$, we have $a \geq 2$ and $b \geq 3$. Therefore, the edges $\{e_0, e_a, e_{2a}, e_b, e_{ab}, e_{(a+1)b}\}$ are distinct. The edges $\{e_0, e_a, e_{2a}, e_{ab}\}$ are pairwise adjacent in G, and so they must induce a copy of $K_{1,4}$. Let u be the vertex common to all four edges. Since the edges $\{e_0, e_b, e_{ab}, e_{(a+1)b}\}$ are pairwise adjacent in G, these four edges must also induce a copy of $K_{1,4}$. It follows that e_b and $e_{(a+1)b}$ must also have vertex

u as one of its endpoints. But then $e_a \sim e_b$, which implies that $b - a \in tS$, a contradiction. Thus, we must have $m = 1$.

If $m = 1$, then $L(G) = C_{ab,tS}$, where the generating set tS includes every element $ka \in tS$ for $1 \leq k \leq \lfloor \frac{b}{2} \rfloor$, and $lb \in tS$ for $1 \leq l \leq \lfloor \frac{a}{2} \rfloor$. First assume that tS contains no other elements. Then this implies that $tS = \{1 \leq k \leq \lfloor \frac{ab}{2} \rfloor : a|k \text{ or } b|k\}$. From Lemma 1.5, this is precisely the line graph for $G = K_{a,b}$, where $\gcd(a, b) = 1$. By Theorem 1.1, $L(G) = C_{ab,tS}$ implies that $G = K_{a,b}$.

Therefore, suppose that tS contains other elements than the multiples of a and b. Let c be the *smallest* element of tS that is not a multiple of a or b. Note that $c > a$ and $c > b$ since (a, b) is the smallest pair with $\gcd(a, b) = 1$, and $a, b \in tS$.

Let $e_0 = xy$ in G. Since each multiple of a is an element of tS, the edges $e_0, e_a, e_{2a}, e_{3a}, \ldots$ all share the same vertex in G. Without loss, assume this vertex is x. Similarly, the edges $e_0, e_b, e_{2b}, e_{3b}, \ldots$ all share the same vertex in G. This common vertex must be y, since $b - a \notin tS$. Now consider e_c, which shares a common vertex with e_0. Without loss, assume e_c has an endpoint x. Then, e_c is adjacent to e_{ka} for all $k \geq 1$, where the index is reduced mod n. Thus, $|c - ka|_n = |c - ka|_{ab}$ is an element of tS for all $k \geq 1$.

Let $c = pa + q$, where (p, q) is the unique integer pair with $0 \leq q \leq a - 1$. Letting $k = p$ and $k = p + 1$, we have $|c - pa|_n = q \in tS$ and $|c - (p+1)a|_n = a - q \in tS$. By the minimality of c, both q and $a - q$ must be multiples of a or b. Clearly neither is a multiple of a. Thus, q and $a - q$ must both be multiples of b. But then its sum, $q + (a - q) = a$, must be a multiple of b. This contradicts the fact that $\gcd(a, b) = 1$.

We have shown that if tS contains some element c other than multiples of a or b, we obtain a contradiction. Thus, tS cannot contain any other elements than the multiples of a and b. We have proven that if $L(G) = C_{n,tS}$ is a circulant, then we must have $n = ab$ and $tS = \{1 \leq k \leq \lfloor \frac{ab}{2} \rfloor : a|k \text{ or } b|k\}$. From our earlier analysis, $L(G) = C_{ab,tS}$ implies that $G = K_{a,b}$. This completes the proof of Theorem 2.2. $\square$

Combining Theorems 2.1 and 2.2, we have proven that if $L(G)$ is a circulant, then G *must* be one of K_4, C_n, or $K_{a,b}$ where $\gcd(a, b) = 1$. We have now given a complete characterization of all circulant line graphs. This completes the proof of Theorem 1.6. $\square$

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