MATH/CSCI 2113

Solutions to assigment 3

1. Give the lexicographically smallest derangement of $1, 2, 3, 4, 5$ and its immediate successor.

   21453 and 21543

2. Ms. Pezzulo teaches geometry and then biology to a class of 12 students in a classroom that has exactly 12 desks. In how many ways can she assign the students to these desks so that (a) no student is seated at the same desk in both classes?

$$12!d_{12}$$

   (see formula in book for d_{12}, the number of derangements of 12 elements.)

   (b) there are exactly 6 students that have the same desk for both classes.

$$12!\binom{12}{6}d_6$$

3. In how many ways can five couples be seated around a circular table so that no couple is sitting next to each other? (Assume there is not distinction between two arrangements that can be obtained from each other by rotating the table.)

   Let c_i be the property that couple i sits together ($i = 1, 2, \ldots, 5$).

   $N(c_i) = 2 \cdot 8!$, $N(c_i, c_j) = 2^2 \cdot 7!$, $N(c_i, c_j, c_k) = 2^3 \cdot 6!$, $N(c_i, c_j, c_k, c_\ell) = 2^4 \cdot 5!$, and $N(c_1, c_2, c_3, c_4, c_5) = 2^5 \cdot 4!$.

   So the answer is:

$$9! - 5 \cdot 2 \cdot 8! + \binom{5}{2} 2^2 \cdot 7! - \binom{5}{3} 2^3 \cdot 6! + 5 \cdot 2^4 \cdot 5! - 2^5 \cdot 4!.$$

4. Find a generating function used for finding the number of ways of selecting r balls out of

- 4 red, 3 blue, 6 orange and 2 green balls.

  $$(1+x+x^2+x^3+x^4)(1+x+x^2+x^3)(1+x+x^2+x^3+x^4+x^5+x^6)(1+x+x^2)$$

- 2 red, 5 blue, 4 orange and 3 green balls, if at least 1 ball of each kind must be selected.

  $(x+x^2)(x+x^2+x^3+x^4+x^5)(x+x^2+x^3+x^4)(x+x^2+x^3)$

- 6 red, 12 black, 7 white and 10 green balls, if at least 1 ball of each kind must be selected, and an even number of red balls and an odd number of white balls must be selected.

  $$(x^2+x^4+x^6)(x+x^2+x^3+\ldots+x^{12})(x+x^3+x^5+x^7)(1+x+x^2+\ldots+x^{10})$$

5. Find the coefficient of x^{18} and x^{19} in $f(x)=\frac{x^4}{1-x^5}$.

   The coefficient of x^{18} is 0 and the coefficient of x^{19} is 1.

6. Find the first five terms of the sequence generated by each of the following generating functions:

   (a) $\frac{x^2}{1-x^2}=x^2+x^4+x^6+x^8+x^{10}+\ldots$

   (b) $\frac{x^3}{(1-x)^2}=x^3+2x^4+3x^5+4x^6+5x^7+\ldots$

   (c) $\frac{1}{3-x}=\frac{1}{3}+\left(\frac{1}{3}\right)^2x+\left(\frac{1}{3}\right)^3x^2+\left(\frac{1}{3}\right)^4x^3+\left(\frac{1}{3}\right)^5x^4+\ldots$

   (d) $x^3(1-2x)^4=x^3-8x^4+24x^5-32x^6+16x^7$