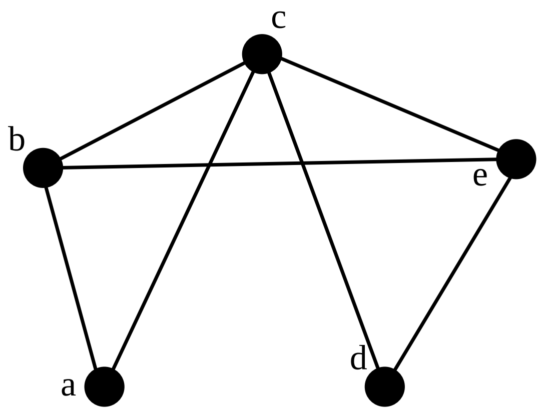
MATH/CSCI 2113
Solutions to assigment 5

1. Consider the graph G, given below. Form the adjacency matrix for this graph, and use this matrix to compute the number of walks of length 4 between all pairs of vertices.



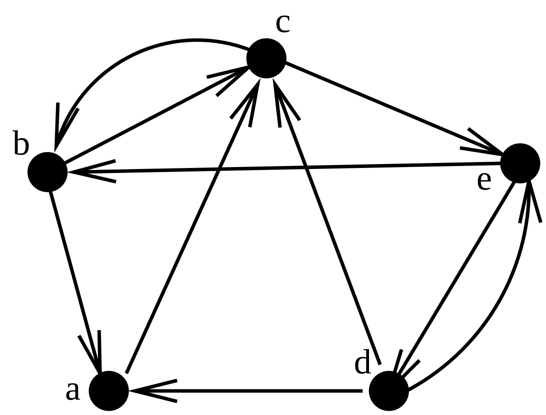
Adjacency matrix (rows and columns corresponding to vertices a, b, c, d, e, in that order):

$$\begin{bmatrix} & 0 & 1 & 1 & 0 & 0 \\ & 1 & 0 & 1 & 0 & 1 \\ A = & 1 & 1 & 0 & 1 & 1 \\ & 0 & 0 & 1 & 0 & 1 \\ & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

The number of walks of length 4 between two nodes, say a and c, is given by the corresponding entry in A^4, so the entry A^4_{13} for the example.

$$A^4 = \begin{bmatrix} 11 & 11 & 13 & 9 & 14 \\ 11 & 19 & 19 & 14 & 14 \\ 13 & 19 & 26 & 13 & 19 \\ 9 & 14 & 13 & 11 & 11 \\ 14 & 14 & 19 & 11 & 19 \end{bmatrix}$$

2. Consider the directed graph of section in the text book. Form the adjacency matrix for this graph, and use this matrix to compute (i) the number of walks of length 3 between all pairs of vertices, (ii) the number of common out-neighbours of each pair of vertices, and (iii) the number of common in-neighbours of each pair of vertices.



Adjacency matrix (rows and columns corresponding to vertices a, b, c, d, e, in that order):

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

(a) The number of (directed) walks of length 3 between two nodes, say a and c, is given by the corresponding entry in A^3, so the entry A^4_{13} for the example.

$$A^3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 3 & 0 & 2 \\ 3 & 2 & 3 & 1 & 2 \\ 0 & 3 & 2 & 1 & 2 \end{bmatrix}$$

(b) The number common out-neighbours of a pair of nodes, say a and c, is given by the corresponding entry in AA^T, so the entry AA^T_{13} for the example.

$$AA^T = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 1 & 1 \\ 1 & 2 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix}$$

(c) The number of common in-neighbours of a pair of nodes, say a and c, is given by the corresponding entry in A^TA, so the entry A^TA_{13} for the example.

$$A^TA = \begin{bmatrix} 2 & 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 & 1 \\ 2 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 2 \end{bmatrix}$$

3. If A is the adjacency matrix of a graph G, then what do the entries of the matrix AA^TA represent? Explain your answer.

$$\begin{aligned} (AA^TA)_{ij} &= \sum_{k=1}^n A_{ik}(A^TA)_{kj} \\ &= \sum_{k=1}^n \sum_{\ell=1}^n A_{ik}A^T_{k\ell}A_{\ell j} \\ &= \sum_{k=1}^n \sum_{\ell=1}^n A_{ik}A_{\ell k}A_{\ell j} \end{aligned}$$

This sum counts exactly the pairs k, ℓ for which $A_{ik} = A_{\ell k} = A_{kj} = 1$ (if any of these entries is zero, then the product is also zero, and contributes nothing to the sum). If $A_{ik} = A_{\ell k} = A_{kj} = 1$, then the graph has edges (i, k), (ℓ, k) and (k, j). So the sum counts the number of paths from i to j which start with a forward edge (i, k), followed by a reverse edge (ℓ, k), followed by a forward edge (k, j).

4. Draw all graphs with vertex set $\{a, b, c, d\}$ and three edges. Indicate which of these graphs are isomorphic.

There are 20 such graphs. They fall into three types: a triangle and an isolated edge, a star shape with one vertex of degree 3 and 3 vertices of degree 1, and a path of length 3.

5. (a) Give an example of a graph so that the graph itself and its complement are both connected. (b) Give an example of a graph where the graph itself is connected, and its complement has two components.

Many answers possible. For example: (a) a cycle with 5 vertices, (b) a cycle with four vertices.

6. (a) Given an example of a graph that is isomorphic to its complement.

For example, a cycle with 5 vertices, or a path with 3 vertices.

(b) Derive a formula (in terms of n, the number of vertices) for the number of edges in a graph that is isomorphic to its complement. Explain your answer.

Let m and m' be the number of edges in the graph and its complement, respectively. Then

$$m + m' = \binom{n}{2} = \frac{1}{2}n(n-1).$$

Since the graph and its complement are isomorphic, they must have the same number of edges, so $m = m'$.

So $2m = \frac{1}{2}n(n-1)$, so $m = \frac{1}{4}n(n-1)$.

7. Do problem 11.3.18 on page 504 of the text book.

   Many answers possible. (a) An Euler circuit must traverse all edges in the graph, hence must have length 19. (b) Just follow your Euler circuit from right after edge $\{d, e\}$ to right before meeting $\{d, e\}$.

8. Do problem 11.5.2 on page 529 of the text book.

   An Euler trail is also a Hamilton path: the graph must be a path (linear tree).

   An Euler circuit is also a Hamilton cycle: the graph must be a cycle.

9. Do problem 11.5.4 on page 529 of the text book.

   (a) The Petersen graph has no Hamilton cycle: Any Hamilton cycle has to cross from the outer five vertices to the inner five. It can do this once, so it uses two of the five edges that connect the inner from the outer group, or it can cross twice, and use four of the five edges. It is straightforward to show (using the symmetry of the Petersen graph), that neither possibility leads to a Hamilton cycle.

Hamilton path: Start at the outer cycle, cross to the inside and complete the inner cycle, then stop.

(b) Remove any vertex. This means that three edges get removed, and three remaining vertices will now have degree 2, so both their incident edges must be part of the Hamilton cycle. That means that you now have determined already 6 edges of the Hamilton cycle. It is easy to complete the cycle at this point. By symmetry, if it works for one vertex, it works for all.