MATH/CSCI 2113

Assigment 6

Solutions

1. Do problem 11.3.28 on page 505 of the book. Explain your answer.

   This problem is assigned as a bonus problem, due Wednesday, April 10.

2. Consider the hypercubes Q_n (see Example 11.12 on page 496 of your text book).

   (a) For what values of n does Q_n have an Euler circuit? Explain your answer.

   The degree of each vertex in Q_n equals n, because a bit string of length n can be changed in n different coordinates. A graph has an Euler circuit if and only if every degree is even. So Q_n has an Euler circuit if and only if n is even.

   (b) Does Q_3 have a Hamilton cycle? If not, explain why not. If so, give the Hamilton cycle.

   Yes. Draw Q_3, and indicate Hamilton cycle.

   (c) For what values of n does Q_n have a Hamilton cycle? Explain your answer.

   For each value of $n \geq 2$. The Hamiltonian cycle can be formed by induction as follows. Base case: Q_2 is a cycle, so it has a Hamiltonian cycle. Induction step: Assume that Q_k has a Hamiltonian cycle. As explained in the book, Q_{k+1} can be obtained from two copies of Q_k by joining the corresponding vertices in each copy. The two Hamiltonian cycles in each copy of Q_k can then also be connected to form a Hamiltonian cycle for Q_{k+1}.

3. Prove the following statement: any connected graph with n vertices and $n-1$ edges is a tree.

   Proof. Let G be a connected graph with n vertices and $n-1$ edges. Proof by contradiction: suppose G is not a tree. Then G must contain cycles. Removing an edge from a cycle will not disconnect G. Remove an edge from each cycle of G, until no cycles remain. The remaining graph G' has less than $n-1$ edges, is still connected, and is acyclic. So G' is a tree. But a tree with n vertices always has $n-1$ edges, while G' has less than $n-1$ edges, so this leads to a contradiction. QED

4. Prove the following statement: any acyclic graph with n vertices and $n-1$ edges is connected.

   Proof by contradiction. Suppose G is an acyclic graph with n vertices and $n-1$ edges, and G is not connected. Then G consists of a number of components, $G_1, \ldots, G_k$. Each component is connected (by definition), and acyclic, so each component is a tree. Let n_i be the number of vertices of component G_i. Then $n_1 + \ldots + n_k = n$. Moreover, component G_i is a tree, so it has $n_i - 1$ edges (for every i). So the number of edges in G equals $(n_1 - 1) + \ldots + (n_k - 1) = (n_1 + \ldots + n_k) - 1 - \ldots - 1 = n - k$. Since G has at least 2 components, $k \geq 2$, so G has at most $n-2$ edges. This is a contradiction with the fact that G has $n-1$ edges. QED

5. Use **strong induction** to prove the following statement: every tree with at least 2 vertices has at least two pendant vertices.

   Proof, by strong induction on n, the number of vertices:
   Basic step: $n = 2$. In this case, the tree consists of just one edge, and both its endpoints are pendant vertices.

   Induction step: Pick an integer $k \geq 3$. Suppose that the statement is true for all tree with *less than* k vertices. Let T be a tree with k vertices. Remove any edge uv of T. This will separate the tree into two smaller trees T_1 and T_2. Since T_1 and T_2 both have less than k vertices, they each have at least two pendant vertices (by the induction hypothesis). So the graph with components T_1 and T_2 has at least four pendant vertices. Putting the edge uv can remove at most two of those pendant vertices, so at least two more are left. Hence, T also has at least two pendant vertices.

6. Consider the two graphs in Figure 12.6 on page 552 of the text book. (a) Determine how many different spanning trees each of these graphs has (you do not need to draw them).

   Graph 1: 6 spanning trees (you can remove any one edge from the cycle)
   Graph 2: $6^2 = 36$ spanning trees (you can remove any one edge from one cycle AND from the other cycle)

   (b) For graph (1), how many non-isomorphic spanning trees are there? Draw one of each type.

Three.

7. Do problem 12.2.6 on page 569 of the text book.

   Preorder: 1,2,5,9,14,15,10,16,17,3,6,4,7,8,11,12,13

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