MATH/CSCI 2113

Assigment 8 — Solutions

1. Do problems 7.1.5 and 7.1.6 on page 318 of the text book.

(a) Reflexive, antisymmetric, transitive. Partial order.
(a) Reflexive, transitive. Not antisymmetric, because $2|-2$ and $-2|2$, for example.
(c) Reflexive, symmetric, transitive. Equivalence relation.
(d) Symmetric.
(e) Symmetric.
(f) Reflexive, symmetric, transitive. Equivalence relation.
(g) Reflexive, symmetric.
(h) Reflexive, antisymmetric, transitive. Equivalence relation.

2. Do problem 7.1.10 on page 318 of the text book.

Note first: $|A \times A| = 4 \cdot 4 = 16$, so $A \times A$ has 2^{16} subsets. So there are 2^{16} different relations on A.

(a) A reflexive relation must contain $(w, w), (x, x), (y, y), (z, z)$. There are $16 - 4 = 12$ remaining elements in $A \times A$. So there are 2^{12} ways to complete the relation.

(b) There are $\binom{4}{2} = \frac{4 \cdot 2}{2} = 6$ pairs (a, b) in $A \times A$ with $a < b$. If any of these is in a relation, and the relation is symmetric, then (b, a) is also in the relation. In addition, there are 4 element of the type (a, a). Each of these can be either in or not in the relation. So there are 2^{10} different relations.

(c) 2^6.

(d) 2^{11}.

(e) 2^9.

(f) For each pair $(a, b) \in A \times A$ with $a < b$, we can choose either (a, b) or (b, a) or neither, but not both. There are 6 such pairs (see (b)), so there are 3^6 choices.

The pairs of type (a, a) can be either in or not in the relation, this gives 2^4 choices. Total: $3^6 2^4$ relations.

(g) $3^5 2^4$.

(h) A relation that is symmetric and antisymmetric can only contain pairs of type (a, a). There are 4 such pairs, so 2^4 relations.

(i) Exactly one.

3. Do problem 7.2.18 on page 330 of the text book.

   (a) The relation is:

   $$R = \{(v, w), (v, x), (w, v), (w, x), (w, y), (w, z), (x, z), (y, z)\}.$$

   Drawing the graph is straightforward.

   (b) Similar.

4. Do problem 7.2.26 on page 330 of the text book.

   The graph of R^n can be obtained as follows: there is an edge between to nodes if and only if there is a walk of length n between the nodes in the original graph. If the graph was only a 4-cycle, then $R^5 = R$, because between any two adjacent nodes on the cycle there is also a walk of length 5: first go once around the cycle. In fact $R^{4k+1} = R$ for every integer k. Similary, if the graph was only a 3-cycle, then $R^{3\ell+1} = R$ for every integer ℓ. For this graph, which contains of a 4-cycle and a 3-cycle, we are looking for the smalles n so that $n = 4k + 1$ and $n = 3\ell + 1$ for some integers k, ℓ. Since the least common multiple of 3 and 4 is 12, we get that the smallest such n is $n = 13$.

5. Do problem 7.3.18 on page 340 of the text book.

   (a) Note: Any subset of U with the elements $1, 2, 3$ in it is an upper bound of B.
   (i) 1 (ii) 4 (iii) $\binom{4}{2} = 6$

(b) To elements $1, 2, 3$, add any of the elements $4, 5, 6, 7$. Total: 2^4 upper bounds.

(c) $\{1, 2, 3\}$.

(d) One, $\emptyset$.

(e) $\emptyset$.

6. Do problem 7.3.20 on page 340 of the text book.

   (a) Note: below a general proof is given. However, this particular relation only has four elements, so you can also prove that it is a partial order by inspection, i.e. by drawing the graph, the Hasse diagram, or examining the matrix.

   Proof.:
   Reflexive: Take $(a, b) \in A$. Then $a = a$ and $b \leq b$, so $(a, b)R(a, b)$.

   Antisymmetric. Take two elements (a, b) and (c, d) in A, and suppose that $(a, b)R(c, d)$ and $(c, d)R(a, b)$. Then $a < c$ or $a = c$ and $b \leq d$, and $c < a$ or $c = a$ and $d \leq b$. Since $a < c$ and $c < a$ leads to a contradiction, we must have that $a = c$, and $b \leq d$ and $d \leq b$. So $b = d$, and thus $(a, b) = (c, d)$.

   Transitive: Take three elements (a, b), (c, d) and (e, d) in A, and suppose $(a, b)R(c, d)$ and $(c, d)R(e, f)$. Then $a < c$ or $a = c$ and $b \leq d$, and $c < e$ or $c = d$ and $d \leq f$. In particular, $a \leq c$ and $c \leq e$. If $a < c$ or $c < e$, then $a < e$, so $(a, b)R(e, f)$. If $a = c$ and $c = e$, then $b \leq d$ and $d \leq f$. So $a = e$ and $b \leq f$, so $(a, b)R(e, f)$.

   The next three are easy if you draw the Hasse diagram.

   (b) Minimal: $(0, 0)$. Maximal: $(1, 1)$.

   (c) Least: $(0, 0)$. Maximal: $(1, 1)$.

   (d) This is a total order.

7. Do problem 7.4.8 on page 345 of the text book.

   (a) Reflexive: Pick any $x \in A$. Then $x - x = 0$, and 0 is divisible by 3, so $(x, x) \in R$.

   Symmetric: Pick any $x, y \in A$, and assume that $(x, y) \in R$. Then $x - y$ is divisible by 3. So also $-(x - y)$ is divisible by 3. But $-(x - y) = y - x$, so $y - x$ is divisible by 3, so $(y, x) \in R$.

   Transitive: Pick three elements $x, y, z \in A$, and assume that $(x, y) \in R$ and $(y, z) \in R$. Then $x - y$ and $y - z$ are both divisible by 3. So $x - y = 3k$ for some integer k, and $y - z = 3\ell$ for some integer ℓ. So $x - z = (x - y) + (y - z) = 3k + 3\ell = 3(k + \ell$ is divisible by 3, so $(x, z) \in R$.

   (b) $[1] = \{1, 4, 7\}$, $[2] = \{2, 5\}$, $[3] = \{3, 6\}$.

8. Do problem 7.4.14 on page 346 of the text book.

   (a) Does not exist, because an equivalence relation is reflexive, so must contain all pairs (i, i) for $i = 1, 2, \ldots, 7$.

   (b) $\{(1, 1), (2, 2), (3, 3), \ldots, (7, 7)\}$.

   (c) Does not exist. An equivalence relation must contain all seven pairs (i, i), and as soon as one pair (i, j) is added with $i \neq j$, then (j, i) must also be included, which would lead to 9 elements.

   (d) $\{(1, 1), \ldots, (7, 7), (1, 1), (2, 1)\}$ (different solutions possible).

   (e) $\{(1, 1), \ldots, (7, 7), (1, 1), (2, 1), (3, 4), (4, 3)\}$ (different solutions possible). 5 equivalence classes, of 3 of size 1, 2 of size 2.

   (f) Does not exist. An equivalence relation must contain all seven pairs (i, i), and as soon as one pair (i, j) is added with $i \neq j$, then (j, i) must also be included. So there must be an even number of additional elements. Hence the total number must be odd.

For (g) and (i) consider the following: each equivalence relation partitions A into sets of size $n_1, \ldots, n_k$, where $n_1 + \ldots + n_k = 7$. Since every ordered pair consisting of elements taken from the same part of the partition must be in the relation, the size of such a relation is: $n_1^2 + n_2^2 + \ldots + n_k^2$.

(g) Does not exist. By inspection, there do not exist any numbers $n_1, \ldots n_k$ so that $n_1 + \ldots n_k = 7$, $n_1^2 + \ldots + n_k^2 = 23$. (Note that the largest part of the partition would have to have size at most 4)

(h) Does not exist. See (f)

(i) Does not exist. By inspection, there do not exist any numbers $n_1, \ldots n_k$ so that $n_1 + \ldots n_k = 7$, $n_1^2 + \ldots + n_k^2 = 31$.