Fibonacci numbers

A case study in the use of generating functions to find a direct formula for a recursively defined sequence

Recursive definition:

$$f_0 = 0$$
$$f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \ \textbf{ for all } n \geq 2$$

Step 1: derive an expression for $f(x)$ in terms of $xf(x)$, $x^2f(x)$ and possibly other functions like $\frac{1}{1-x}$, using the recurrence relation.

$$
\begin{aligned}
f(x) &= f_0 + f_1x + f_2x^2 + f_3x^3 + f_4x^4 + \dots \\
&= 0 + x + (f_0 + f_1)x^2 + (f_1 + f_2)x^3 + (f_2 + f_3)x^4 + \dots \\
&= x + f_0x^2 + f_1x^2 + f_1x^3 + f_2x^3 + f_2x^4 + f_3x^4 + \dots \\
&= x + xf(x) - f_0x + x^2f(x) \\
&= x + xf(x) + x^2f(x)
\end{aligned}
$$

Step 2: derive a formula for $f(x)$. (Usually this formula will be a quotient of two polynomials.)

We derived that $(1 - x - x^2)f(x) = x$**, so**

$$f(x) = \frac{x}{(1 - x - x^2)}.$$

Step 3: Try to write the formula for $f(x)$ in terms of functions for which you know the series. Often, you can use partial fractions to split the formulat in terms of the form $\frac{A}{1-bx}$.

The roots of $1-x-x^2$ can be found using the abc-formula. Here:

$$1-x-x^2 = -(x-r_1)(x-r_2),$$

where $r_1 = \frac{-1+\sqrt{5}}{2}$ and $r_2 = \frac{-1-\sqrt{5}}{2}$.

Partial fractions:

$$f(x) = \frac{-x}{(x-r_1)(x-r_2)} = \frac{A}{x-r_1} + \frac{B}{x-r_2},$$

so

$$(x-r_2)A + (x-r_1)B = -x.$$

Plug in $x = r_1$ and $x = r_2$ in this equation to obtain expressions for A and B:

$$A = \frac{-r_1}{r_1 - r_2} = -\frac{r_1}{\sqrt{5}} \textbf{ and } B = \frac{-r_2}{r_2 - r_1} = \frac{r_2}{\sqrt{5}}.$$

So

$$\frac{-\frac{r_1}{\sqrt{5}}}{x-r_1} + \frac{\frac{r_2}{\sqrt{5}}}{x-r_2} = \frac{\frac{1}{\sqrt{5}}}{1-\frac{x}{r_1}} + \frac{-\frac{1}{\sqrt{5}}}{1-\frac{x}{r_2}}.$$

Step 4: use the series expansion of the parts that make up the formula for $f(x)$ to derive a direct formula for f_n for all $n \geq 0$.

$$\begin{aligned} f(x) = &\ \tfrac{1}{\sqrt{5}}\left(1 + \left(\tfrac{x}{r_1}\right) + \left(\tfrac{x}{r_1}\right)^2 + \left(\tfrac{x}{r_1}\right)^3 + \ldots\right) \\ &\qquad -\tfrac{1}{\sqrt{5}}\left(1 + \left(\tfrac{x}{r_2}\right) + \left(\tfrac{x}{r_2}\right)^2 + \left(\tfrac{x}{r_2}\right)^3 + \ldots\right) \\ = &\ \tfrac{1}{\sqrt{5}}\left(1 + \left(\tfrac{1}{r_1}\right)x + \left(\tfrac{1}{r_1}\right)^2 x^2 + \left(\tfrac{1}{r_1}\right)^3 x^3 + \ldots\right) \\ &\qquad -\tfrac{1}{\sqrt{5}}\left(1 + \left(\tfrac{1}{r_2}\right)x + \left(\tfrac{1}{r_2}\right)^2 x^2 + \left(\tfrac{1}{r_2}\right)^3 x^3 + \ldots\right). \end{aligned}$$

So for all $n \geq 0$

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1}{r_1}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1}{r_2}\right)^n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$