Self-test

$a_r=$the number of ways to choose r books from a pile of 7 different Harry Potter books, and 5 identical copies of *The hobbit*.

What is the generating function of a_r?

Recurrence relations

Towers of Hanoi:

h_n = the number of moves required to move n disks from peg to peg in the "Towers of Hanoi" game.

$$h_1 = 1$$
$$h_n = 2h_{n-1} + 1 \ \textbf{ for all } n \geq 2$$

$$h(x) = \sum_{i=1}^{\infty} h_i x^i.$$

$$h(x) = \frac{x}{(1-2x)(1-x)} = \frac{1}{1-2x} - \frac{1}{1-x}$$

Fibonacci numbers

$$f_0 = 0$$
$$f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \ \text{ for all } n \geq 2$$

$$f(x) = \sum_{i=1}^{\infty} f_i x^i.$$

$$f(x) = \frac{x+1}{(1-x-x^2)} = \frac{-1-x}{(x-r_1)(x-r_2)},$$

where $r_1 = \dfrac{-1+\sqrt{5}}{2}$ **and** $r_2 = \dfrac{-1-\sqrt{5}}{2}.$

Partial fractions:

$$f(x) = \frac{A}{x - r_1} + \frac{B}{x - r_2} = \frac{\frac{A}{r_1}}{1 - \frac{x}{r_1}} - \frac{\frac{B}{r_2}}{1 - \frac{x}{r_2}},$$

where $A = \dfrac{r_1 + 1}{r_2 - r_1} = \dfrac{1 + \sqrt{5}}{-2\sqrt{5}}$, **so** $\dfrac{A}{r_1} = \dfrac{1}{\sqrt{5}}$, **and**

(by a similar calculation), $\dfrac{B}{r_2} = -\dfrac{1}{\sqrt{5}}$.

So for all $n \geq 0$

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1}{r_1}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1}{r_2}\right)^n = \frac{1}{\sqrt{5}}\left[\left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n\right].$$