Topics in Graph Theory – 5
January 28 and 30, February 4, 2014.

List colourings

A common requirement in real-life applications of graph colouring is that the "colours" available are limited by external considerations. This concept gives rise to the concept of list colouring.

Given a graph $G = (V, E)$ and a colour set A, a *list assignment* is a function $L : V \to \mathcal{P}(A)$. The set $L(v)$ is called the *list* at vertex v, and represents the set of possible colours at v. A *list colouring* for a given graph G and list assignment L is a proper colouring f of G such that, for all $v \in V$, $f(v) \in L(v)$. A graph G is k-choosable or k-list colourable if, for every list assignment such that, for all $v \in V$, $|L(v)| = k$, there exists a list colouring of (G, L). The *list chromatic number* $\chi_\ell(G)$ of G is the least k so that G is k-choosable.

One possible list assignment is to give each vertex the same list. In this case, the problem reverts to the regular colouring problem, and a list colouring exists precisely when the graph has chromatic number at least as large as the size of the common list. Thus, for every graph G, $\chi_\ell(G) \geq \chi(G)$. We will see in a presentation in class that there exist bipartite graphs with arbitrary large list chromatic number, so the gap between $\chi_\ell(G)$ and χ_G can be arbitrarily large.

Given a list assignment, we can also employ the greedy colouring algorithm to find a list colouring. As before, vertices are coloured in predetermined order. At each vertex v, a colour in $L(v)$ is chosen which does not appear on any of the coloured neighbours. Clearly, if $L(v)$ is of larger size than the number of coloured neighbours, such a colour exists. Therefore, if we have a greedy ordering $v_1, v_2, \ldots, v_n$ such that, for every vertex v_i, $|N(v_i) \cap \{v_1, v_2, \ldots, v_{i-1}\}| \leq k$, then the graph is $(k+1)$-choosable. For example, for the graph in assignment 1 formed by intersecting lines in the plane, the vertices could be ordered so that each vertex has at most two coloured neighbours. Thus the list-chromatic number is at most 3. Similarly, if we have a perfect elimination ordering, then each vertex has at most $\omega - 1$ coloured neighbours, so $\chi_\ell(G) = \chi(G) = \omega(G)$.

We have seen that there are many connections between colourings and orientations. Here is one more. First we need some definition. Given an orientation of a graph, the *out-degree* of a vertex v, notation $\deg^+(v)$, is the number of edges that have v as their tail. A *kernel* is an independent set A in G so that each vertex $v \in V(G)$ is either in a, or is the tail of an edge with head in A.

Theorem 1. *Let $G = (V, E)$ be a graph. If G has an orientation such that every induced subgraph has a kernel, and $L : V \to C$ is a list assignment for V so that, for all $v \in V$, $|L(v)| \geq \deg^+(v) + 1$, then there exists a list colouring of (G, L).*

Proof. The proof is by induction on the total number of edges of G. If G has no edges, then G satisfies the condition trivially, and each vertex has out-degree 0. So for any assignment of lists of size at least 1, a list-colouring can be found. Fix a colour $c \in C$, and let G_c be the subgraph by all vertices whose list contains colour c. By assumption, G_c has a kernel, say A. Assign colour c to all vertices of A.

Now consider $H = G - A$, and let L_H be a list assignment for H obtained by removing colour v: $L_H(v) = L(v) - \{c\}$ for all $v \in V - A$. Now for each v of $G_c - A$, we have that $\deg^+_H(v) = \deg^+(v) - 1$, where $\deg^+_H(V)$ is the out-degree of v in H. On the other hand, $|L_H(v)| = |L(v)| - 1$. So $|L_H(v)| \geq \deg^+_H(v) + 1$. For vertices in H which are not in G_c, so whose list does not contain v, $\deg^+_H(v) \leq \deg^+(v) \leq |L_H(v)| - 1$.

Thus H satisfies the conditions, so by induction there exists a list colouring of (H, L_H). Adding the vertices in A, coloured with colour c, makes this into a list colouring for (G, L).

□

The concept of list colouring can be equally applied to edge colourings. Thus, $\chi'_\ell(G)$ is the *list chromatic index* of G, and is the minimum number k such that, for any assignment of lists of size k to the edges of G, a list colouring can always be found.

In general, edge colourings are "nicer" than vertex colourings. For example, we have the theorem that, for all bipartite graphs G, $\chi'(G) = \Delta(G)$. In fact, for simple graphs G, we have that $\chi'(G) \leq \Delta + 1$. (Proof of this theorem skipped in this class, but worth looking up!) This led Vizing to the following conjecture.

[Vizing] For all graphs G, $\chi'_\ell(G) = \chi'(G)$.

The conjecture was proved for bipartite graphs. For bipartite graphs, the line graph has an obvious representation. Each edge x_i, y_j in a graph with bipartition X, Y can be represented as a subsquare in an $|X| \times |Y|$

square, where each row corresponds to an element of X, and each column to an element of Y. Subsquares are connected iff they are in the same row and column. It turns out that, for a specific orientation of the line graph, a kernel can be found using the concept of stable matchings.

Given sets $X = \{x_1, x_2, \ldots, x_3\}$ and $Y = \{y_1.y_2, \ldots, y_n\}$, as well as a ranking of the elements of Y for each element of X, and a ranking of the elements of X for each element of Y, a *stable matching* is a subset M of $X \times Y$ such that each element of X and each element of Y occurs exactly once in M (so M is a perfect matching), and, for every pair (x, y) not in M, the following holds. Let x' be the unique element of X matched to y, and y' the unique element of Y matched to x. (So (x, y') and (x', y) are in M). Then x prefers y' to y or y prefers x' to x.

Gale and Shapley showed that a stable matching always exists, no matter how the rankings are, and they gave an algorithm to find such a matching.

Lemma 2. *If G is a bipartite graph with maximum degree Δ, then $L(G)$ has an orientation with the property that each subgraph of $L(G)$ has a kernel, and each vertex of $L(G)$ has out-degree at most $\Delta - 1$.*

Proof. Assume *wlog* that $|X| = |Y| = n$. (If not, add isolated vertices.) Let $k = \Delta(G)$. Let $f : X \times Y \to [k]$ be a vertex colouring of $L(G)$ with k colours. By König's theorem, such a colouring exists. Now orient the edges of $L(G)$ as follows. Horizontally, orient edges from larger colours to smaller colours. So if $c(x, y) > c(x, y')$ then the edge is oriented from (x, y) to (x, y'). Vertically, orient edges from smaller colours to larger colours, so if $c(x, y) > c(x', y)$ then the edge is oriented from (x', y) to (x, y).

Note that each vertex (x, y) has out-degree at most $k - 1$. Namely, Let $c(x, y) = i$. Then any horizontal edge from (x, y) to a vertex (x, y') must go to a vertex of colour in $\{1, 2, \ldots, i - 1\}$, while any vertical edge must go to a vertex (x', y) of colour in $\{i + 1, \ldots, k\}$. Since each colour can occur at most once in any row or column, this implies that any vertex has out-degree at most $k - 1$.

Now consider any induced subgraph H of $L(G)$. Form the following preference lists. Each vertex x ranks the vertices in Y as follows. First, elements $y \in Y$ so that (x, y) is in H are ranked, in *increasing* order of the colour of the pair (x, y). Then, the other elements of Y are ranked in any arbitrary order, but all being of less preference than the first set. Each vertex y ranks the vertices in X as follows. First, elements $x \in X$ so that (x, y) is in H are ranked in *decreasing* order of the colour of the pair (x, y). Then, the other elements of X are ranked in any arbitrary order, but of less preference than the first set.

Let M be a stable matching for these preference rankings. We claim that the set A consisting of all pairs (x, y) in M that correspond to vertices in H forms a kernel in H with the given orientation. Let (x, y) be a pair occurring in H which is not in A. Let $x' \in X$ and $y' \in Y$ be so that (x, y') and (x', y) are in M (such elements must exist). By the definition of a stable matching, x prefers y' to y or y prefers x' to x. If x prefers y' to y, then the pair (x, y') must be in H, and $c(x, y') < c(x, y)$. Therefore, $(x, y') \in A$ (by definition of the ranking) and there is an edge directed from (x, y) to (x, y'). If y prefers x' to x, then the pair (x', y) must be in H, and $c(x, y) < c(x', y)$. Therefore, $(x', y) \in A$ and there is an edge directed from (x, y) to (x', y). Therefore, (x, y) is the tail of at least one edge with head in A. Thus, A is a kernel. $\square$

The proof of the following theorem now follows from the previous lemma and theorem.

Theorem 3 (Galvin, '94)**.** *For all bipartite graphs G, $\chi'_\ell(G) = \chi'(G)$.*