Hall's Condition

Given a collection of sets $A_1, \dots, A_n$, a *System of Distinct Representatives (SDR)* is a collection of distinct elements $x_1, \dots, x_n$ so that, for $1 \le i \le n$, $x_i \in A_i$.

Given an index set $J \subseteq \{1, 2, \dots, n\}$,

$$A(J) = \bigcup_{j \in J} A_j,$$

so $A(J)$ is the union of all sets whose index is in J.

A collection of sets $A_1, \dots, A_n$ satisfies *Hall's Condition (HC)* if,

$$\text{for every index set } J \subseteq \{1, 2, \dots, n\}, \quad |A(J)| \ge |J|.$$

Hall's Theorem

Hall's Theorem: For every collection of sets collection of sets $A_1, \ldots, A_n$, there exists a System of Distinct Representatives if and only if Hall's Condition holds.

Proof. SDR$\Rightarrow$ HC. Let $x_1, \ldots, x_n$ be an SDR for the collection of sets $A_1, \ldots, A_n$. Fix $J \subseteq \{1, 2, \ldots, n\}$, and let

$$S = \{x_j : j \in J\}.$$

Since all x_i are distinct, $|S| = |J|$. By definition of an SDR, $S \subseteq A(J)$, so $|S| \leq |A(J)|$. Thus $|J| \leq |A(J)|$. $\square$

Hall's Theorem: proof of sufficiency

Hall's Theorem: For every collection of sets collection of sets $A_1, \ldots, A_n$, there exists a System of Distinct Representatives if and only if Hall's Condition holds.

Proof. HC$\Rightarrow$ SDR. Induction on n.

Base case: $n = 1$. HC implies that $|A_1| \geq 1$, so $A_1 \neq \emptyset$. Thus we can choose $x_1 \in A_1$, which is an SDR.

Induction step. Fix $n > 1$. Suppose HC$\Rightarrow$ SDR for all collections of less than n sets. Let $A_1, \ldots, A_n$ be a collection of sets for which HC holds. We distinguish two cases.

Case 1: For all non-empty $J \subset \{1, 2, \ldots, n\}$, $|A(J)| > |J|$.

Case 2: There exists non-empty $J \subset \{1, 2, \ldots, n\}$ so that $|A(J) = |J|$.

Hall's Theorem: proof of sufficiency

Case 1: For all non-empty $J \subset \{1, 2, \ldots, n\}$, $|A(J)| > |J|$. Pick $x_n \in A_n$. (A_n is not empty because of HC applied to $J = \{n\}$).

Remove x_n from all other sets. Formally, let $A'_j = A_j - \{x_n\}$ for all $1 \leq j \leq n-1$.

Claim: $A'_1, \ldots, A'_{n-1}$ satisfies HC.

Proof of claim: take $J \subseteq \{1, 2, \ldots, n-1\}$. Then $A'(J) = A(J) - \{x_n\}$, so $|A'(J)| \geq |A(J)| - 1$. By assumption of this case, $|A(J)| \geq |J| + 1$. Thus $|A'(J)| \geq |J|$. $\square$.

Therefore, an SDR for $A'_1, \ldots, A'_{n-1}$ exists, by the induction hypothesis, and none of the representatives equals x_n. Adding x_n gives an SDR for the original collection.

Hall's Theorem: proof of sufficiency

Case 2: There exists non-empty $J \subset \{1, 2, \ldots, n\}$ so that $|A(J) = |J|$. Let $\bar{J} = \{1, 2, \ldots, n\} - J$.

The collection of sets A_j, $j \in J$ satisfies Hall's condition, and, since J is a strict subset of $\{1, 2, \ldots, n\}$, the collection contains less than n sets. So, by the induction hypothesis, we can find an SDR for this collection. Note that all elements in $A(J)$ must be part of this SDR.

Remove the elements of this SDR from all remaining sets. Formally, for all $j \in \bar{J}$, let $A'_j = A_j - A(J)$.

Claim: The collection of sets A'_j, $j \in \bar{J}$ satisfies HC.

Assuming the claim, by the induction hypothesis there exists an SDR for A'_j, $j \in \bar{J}$. By construction, this SDR does not contain any elements from $A(J)$. Combining the two SDRs gives an SDR for the original collection.

Hall's Theorem: proof of sufficiency

Case 2: There exists non-empty $J \subset \{1, 2, \ldots, n\}$ so that $|A(J) = |J|$. Let $\bar{J} = \{1, 2, \ldots, n\} - J$.

For all $j \in \bar{J}$, let $A'_j = A_j - A(J)$.

Claim: The collection of sets A'_j, $j \in \bar{J}$ satisfies HC.

Proof of claim: take $K \subseteq \bar{J}$. Suppose by contradiction that $|A'(K)| < |K|$.

Consider $A(K \cup J)$. Note that

$$A(K) \subseteq A'(K) \cup A(J), \text{ so } A(K \cup J) = A(K) \cup A(J) = A'(K) \cup A(J).$$

Since $A'(K)$ and $A(J)$ are disjoint, we have

$$|A(K \cup J)| \leq |A'(K)| + |A(J)| < |K| + |J|,$$

since $|A(J)| = |J|$ by assumption of this case. This contradicts the original assumption that HC holds for $A_1, \ldots, A_n$. $\square$