MATH 4320/5320
Combinatorial Optimization
Assignment 2

Due Thursday, Jan. 22

1. Prove Hall's theorem about the existence of a matching in a bipartite graph covering one side of the bipartition.

2. A k-regular graph is a graph where each node has degree k.

   (a) Prove that each bipartite k-regular graph ($k \geq 1$) has a perfect matching. *Hint: use vertex covers.*

   (b) An *edge colouring* of a graph is an assignment of colours to the edges of a graph so that no two edges that share and endpoint receive the same colour. Use (a) to prove that each bipartite k-regular graph has an edge-colouring using exactly k colours. *Note: in any edge colouring, the set of all edges of one colour must be a matching.*

3. (5320 students only) Show that the convex hull of the set of perfect matchings in a bipartite graph is given by the system:

$$\text{For all } v \in V, \qquad \sum_{v \text{ endpoint of } e} x_e = 1, \qquad \text{for all } e \in E, \quad x_e \geq 0.$$

   Hint: Consider any rational point x in the polyhedron represented by the inequalities. Consider the components of x to be weights on the edges of the graph. Multiply the weights by a constant until they are integers. Use (a). Note that the theorem given in (a) also works for multigraphs, i.e. graphs that can have more than one edge between any pair of nodes.

Due Thursday, Jan. 29

In each of the following problems, formulate an integer linear program to find the desired combinatorial object (use the inequality descriptions you found last week). Find the dual of the corresponding LP relaxation. Describe the combinatorial objects described by the feasible integer solutions of the dual

1. Find the minimum size of a kernel in a given directed graph. A kernel is an independent set so that each node not in the set has an out-edge pointing to a node in the set.

2. Find the maximum size of a *coclique* (or independent set) in a given graph. Use the inequality system based on the fact that a coclique can contain at most one node of every clique. (A clique is a set of nodes that are all adjacent.)

3. (only for 5320 students) A *doubly independent* set of edges in a graph $G = (V, E)$ is a set of edges in E so that for any two of these edges, they do not share an endpoint, and there is no edge in G connecting their endpoints. (This is also known as an *induced matching*). Find the maximum doubly independent set in G.