What is the perimeter of the largest rectangle that can be inscribed inside a circle of radius R?

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- Find dy/dx from circle equation: $2x + 2y(dy/dx) = 0$, so $dy/dx = -x/y$.
- So at critical point, $4 - 4(x/y) = 0$, so $x = y$.

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- So at critical point, $4 - 4(y/x) = 0$, so $x = y$.
- So $(x, y) = (R/\sqrt{2}, R/\sqrt{2})$, and $P(x, y) = 4\sqrt{2}R$.
- Is this really a maximum? Sketch $P(x) = 4x + 4\sqrt{R^2 - x^2}$.

What is the perimeter of the largest rectangle that can be inscribed inside a circle of radius R?

- Put $x = R\cos\theta$, $y = R\sin\theta$.

- Perimeter $P = 4R\cos\theta + 4R\sin\theta$.

- At maximum, $dP/d\theta$=0.

$$\frac{dP}{d\theta} = -R\sin\theta + R\cos\theta = 0.$$

- So at critical point, $\sin\theta = \cos\theta$, so $\theta = \pi/4$.

- So $(x, y) = (R/\sqrt{2}, R/\sqrt{2})$, and $P(x, y) = 4\sqrt{2}R$.

- To sketch, note that

$$\cos\theta + \sin\theta = \sqrt{2}(\cos\theta\cos\pi/4 + \sin\theta\sin\pi/4) = \sqrt{2}\cos(\theta - \pi/4).$$

Amplitude Phase formula

$$A\cos(\omega t) + B\sin(\omega t) = \sqrt{A^2 + B^2}\cos(\omega t - \phi),$$

where $\tan(\phi) = B/A$.

Verify:

- $\cos(a - b) = \cos a \cos b + \sin a \sin b$.
- Set $a = \omega t$, $b = \phi$.
- Note $\sin\phi = B/\sqrt{A^2 + B^2}$ and $\cos\phi = A/\sqrt{A^2 + B^2}$.

Amplitude Phase formula – Pendulum

- Pendulum with length L, initial angle θ_0, initial velocity v_0.
- Differential equation:

$$\frac{d^2\theta}{dt} = (g/L)\theta.$$

- General solution: $\theta(t) = A\cos\omega t + B\sin\omega t$, where $\omega = \sqrt{g/L}$.
- From initial conditions: $A = \theta_0$, $B = v_0/\omega$.
- From amplitude-phase formula:

$$\theta(t) = \sqrt{\theta_0^2 + v_0^2/\omega^2}\cos(\omega t - \phi),$$

where $\tan\phi = v_0/(\omega\theta_0)$.

What is the point on a curve closest to the origin?

- Curve is given by an equation $f(x, y) = 0$.
- Distance $D = \sqrt{x^2 + y^2}$.
- Set $dD/dx = 0$ to find critical point:

$$\frac{dD}{dx} = \frac{2x + 2y(dy/dx)}{2D} = 0.$$

- Extreme point when $dy/dx = -x/y$. So slope of tangent line is $-y/x$.
- Line through origin and point (x, y) has slope y/x.

At extreme point, line from origin to point is perpendicular to the tangent line to the curve at this point.

A sugar cube dissolves in a cup of hot coffee. What is the relation between change in volume and surface area?

- Length of sides of cube at time t: $x(t)$.
- Volume $V = x^3$, Surface area $A = 6x^2$, so $A = 6V^{2/3}$.
- $dA/dt = 4V^{-1/3} dV/dt = (4/x) dV/dt$

A sugar cube dissolves in a cup of hot coffee. Dissolving rate is proportional to surface area.

- Length of sides of cube at time t: $x(t)$.
- Volume $V = x^3$, Surface area $A = 6x^2$, so $A = 6V^{2/3}$.
- Differential equation:

$$\frac{dV}{dt} = -kA = -k6V^{2/3}.$$

- Not of the general type of Newton cooling etc.

A sugar cube dissolves in a cup of hot coffee. Dissolving rate is proportional to volume.

- Assume initial volume V_0.
- Differential equation:

$$\frac{dV}{dt} = -kV, \quad t = 0 \quad V = V_0.$$

- This is of general type of Newton cooling etc. Try general solution $V(t) = A + Be^{rt}$, where A, B, r to be determined.
- $dV/dt = rBe^{rt} = r(V - A)$.
- In order to satisfy DE, set $r = -k$ and $A = 0$, so $V(t) = Be^{-kt}$.
- Initial condition: $V_0 = V(0) = B$, so $V(t) = V_0 e^{-kt}$.

A sugar cube dissolves in a cup of hot coffee. Dissolving rate is proportional to volume.

- Volume $V(t) = V_0 e^{-kt}$.
- Time constant is $1/k$.
- At times $1/k$, $2/k$, $3/k$ cube has reached approx. 30% ($1/e$), 10% ($1/e^2$), 5% ($1/e^3$) of its original volume.
- If cube is dissolved to half its orginal volume in 10 minutes (600 sec), we can determine k:
  $V(600) = V_0/2$, so $e^{600k} = 2$, so $k = (\ln 2)/600 \approx 1.16 \cdot 10^{-3}$.