Vortex knots on a torus and their stability

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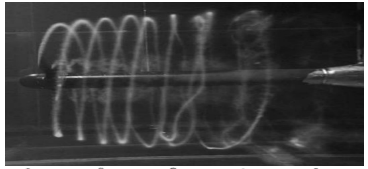
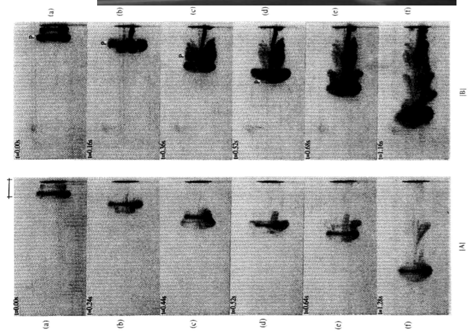
Joint work with Panos Kevrekidis and Chris Ticknor

Examples of vortex lines in 3D:

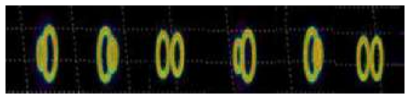
- [Okulov et al 2015]: Wake effect on a uniform flow behind wind-turbine model



- [Lim, 1997]: Formation of turbulent vortex rings table-top experiments



- [Caplan et.al, 2014]: Leapfrogging of vortex rings in a superfluid



Nearly parallel vortex filaments

- Reduced equations for nearly parallel filamens due to [Klein, Majda, Damodaran, 1995], [Lions, Majda, 2000]:

$$-i\frac{\partial}{\partial t}X_k = D\frac{\partial^2}{\partial z^2}X_k + \sum_{j\neq k}\frac{X_k - X_j}{|X_k - X_j|^2}, \quad k = 1\ldots K$$

- Filaments are curves in 3D space parametrized by height z so that, $X_k \in \mathbb{R}^2$ and

$$(\mathrm{Re}(X_k(z,t)), \mathrm{Im}(X_k(z,t)), z) \in \mathbb{R}^3.$$

- This is a generalization of Helmoltz equations of vortex motion (take D=0).
- Vortex lines "diffuse" along the vertical direction; Helmotz interaction along each horizontal slice.
- Possible boundary conditions:

$$\textbf{Case 1: } K \text{ infinite filaments: } z \in \mathbb{R} \tag{1}$$

$$\textbf{Case 2 (periodic BC)}: \;\; X_k(P,t) = X_k(0,t) \tag{2}$$

$$\textbf{Case 3 (connecting filaments):} \;\; X_k(P,t) = X_{k+q}(0,t), \;\; q \in \mathbb{Z}. \tag{3}$$

Co-rotating, spiral vortex filaments

- Reduced equations:

$$-i\frac{\partial}{\partial t}X_k = D\frac{\partial^2}{\partial z^2}X_k + \sum_{j\neq k}\frac{X_k - X_j}{|X_k - X_j|^2},$$

- **Anzatz A:** co-rotating filament steady state:

$$X_k(t,z) = e^{-i\Omega t}\xi_k(z) \tag{4}$$

then ξ_k satisfies

$$0 = D\frac{\partial^2}{\partial z^2}\xi_k - \Omega\xi_k + \sum_{j\neq k}\frac{\xi_k - \xi_j}{|\xi_k - \xi_j|^2}. \tag{5}$$

- **Anzatz B:** spiral state:

$$\xi_k(z) = e^{zi\omega}\hat{\xi}_k, \tag{6}$$

  then $\hat{\xi}_k$ satisfies

$$0 = -\hat{\Omega}\hat{\xi}_k + \sum_{j\neq k}\frac{\hat{\xi}_k - \hat{\xi}_j}{\left|\hat{\xi}_k - \hat{\xi}_j\right|^2} \quad \text{where } \hat{\Omega} = D\omega^2 + \Omega. \tag{7}$$

- Any co-rotating point-vortex configuration gives rise to spiral filaments via $\hat{\Omega} = D\omega^2 + \Omega$.

- **Anzatz C:** ring of vortices:
$$\hat{\xi}_k = re^{i2\pi k/K}.$$
Then r is given by
$$r^2 = \frac{K-1}{2\left(D\omega^2 + \Omega\right)}. \tag{8}$$

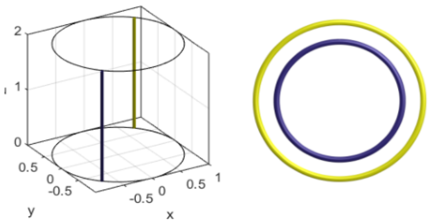
(K,q) toroidal knots

- **Theorem (K,q)-state:** given intergers K, q, let
$$\omega = \frac{2\pi}{P}\frac{q}{K}. \tag{9}$$
Then co-rotating K twisted filaments have the form
$$X_k(t, z) = re^{-i\Omega t}e^{zi\omega}e^{i2\pi k/K} \tag{10}$$
with r as in (8) and it satisfies boundary condition:
$$X_k(P, t) = X_{k+q}(0, t) \tag{11}$$
where the indices are taken modulo K.

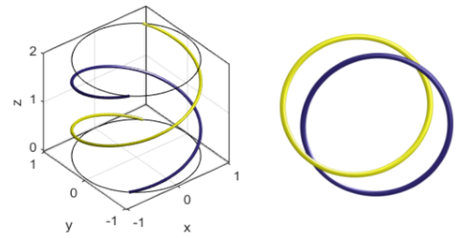
- Such solutions correspond to ***toroidal knots and links***; the distinct number of filaments is given by $\gcd(K, q)$.

- Examples Movies: (2,0) (3,0), (2,2), (3,2) trefoil, (2,3) trefoil

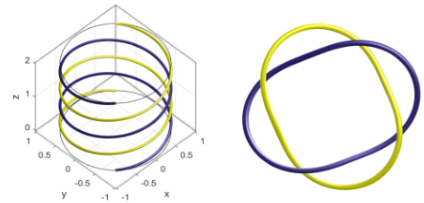
$(K,q) = (2,0)$



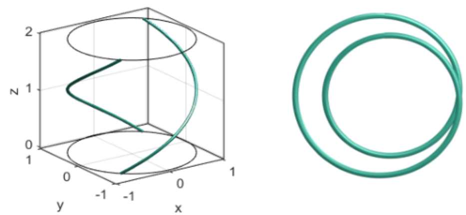
$(K,q) = (2,2)$



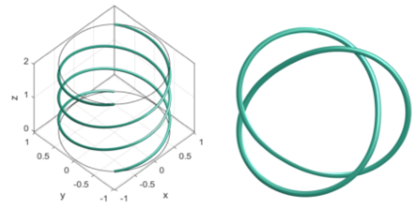
$(K,q) = (2,4)$



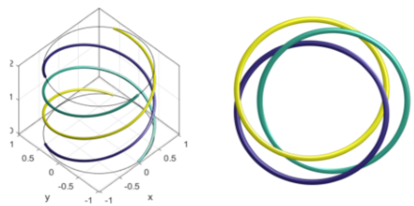
$(K,q) = (2,1)$



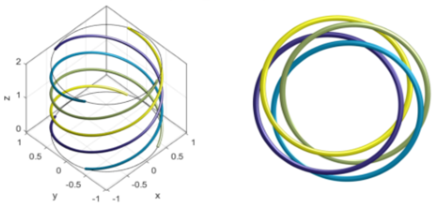
$(K,q) = (2,3)$



$(K,q) = (3,3)$



$(K,q) = (4,4)$



Gallery of toroidal knots

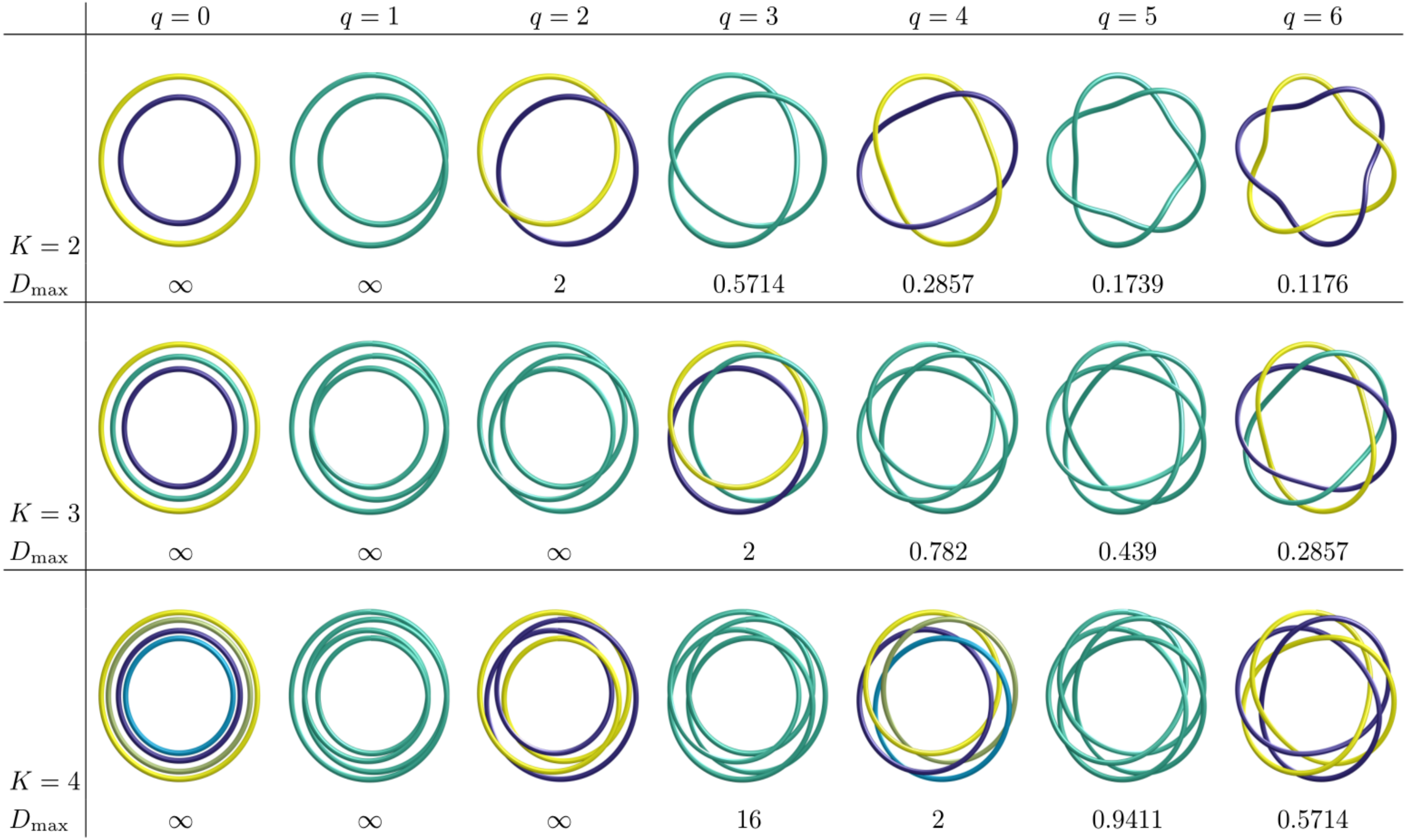


FIG. 3. Stability thresholds for (K, q) steady states with (K, q) as indicated. Here, $P = 2\pi$ and $\Omega = 1$. Since $K \leq 7$, the steady state is stable whenever $0 < D < D_{\max}$. Refer to Theorem 1.2

Stability

- Stability thresholds!

$$-i\frac{\partial}{\partial t}X_k = D\frac{\partial^2}{\partial z^2}X_k + \sum_{j\neq k}\frac{X_k - X_j}{|X_k - X_j|^2}, \quad k = 1\ldots K$$

- Go to rotating frame:

$$X_k(t,z) = e^{-i\Omega t}\xi_k(z,t) \tag{12}$$

then ξ_k satisfies

$$-i\frac{\partial}{\partial t}\xi_k = D\frac{\partial^2}{\partial z^2}\xi_k - \Omega\xi_k + \sum_{j\neq k}\frac{\xi_k - \xi_j}{|\xi_k - \xi_j|^2}. \tag{13}$$

- Imaginary integration: replace $i \to i - \gamma$ and take limit $\gamma \to \infty$. After rescaling

$$\frac{\partial}{\partial t}\xi_k = D\frac{\partial^2}{\partial z^2}\xi_k - \Omega\xi_k + \sum_{j\neq k}\frac{\xi_k - \xi_j}{|\xi_k - \xi_j|^2}. \tag{14}$$

- General perturbation of the (K,q):

$$\xi_k(t,z) = e^{i2\pi k/K}\left(e^{i\omega z}r + h_k(t,z)\right), \quad h_k \ll 1. \tag{15}$$

- **Anzatz:** decompose into fourier modes: m for (x,y) and α for z

$$h_k = e^{i(\alpha+\omega)z} e^{2\pi imk/K} \xi_+(t) + e^{i(-\alpha+\omega)z} e^{-2\pi imk/K} \bar{\xi}_-(t). \tag{16}$$

- yields a 2x2 system:

$$\begin{pmatrix} \hat{\xi}_+ \\ \hat{\xi}_- \end{pmatrix} \lambda = \begin{pmatrix} \left(-D\left(\alpha+\omega\right)^2 - \Omega\right) & \sigma \\ \sigma & \left(-D\left(\alpha-\omega\right)^2 - \Omega\right) \end{pmatrix} \begin{pmatrix} \hat{\xi}_+ \\ \hat{\xi}_- \end{pmatrix}. \tag{17}$$

$$\sigma = \frac{D\omega^2 + \Omega}{K-1} \left(m-1\right)\left(K-m-1\right).$$

$$m = 0 \ldots K-1 \text{ is the mode in (x,y) plane;} \tag{18}$$

$$\alpha = m\omega + M\omega K/q, \quad \text{where } M = 0, 1, 2, 3, \ldots \text{ is the mode in } z \tag{19}$$

- λ satisfies

$$0 = \left(\frac{\lambda}{\Omega}\right)^2 + c\frac{\lambda}{\Omega} + \kappa$$

where $c > 0$ and

$$\kappa := \left(\hat{D}\left(m + MK/q + 1\right)^2 + 1\right)\left(\hat{D}\left(m + MK/q - 1\right)^2 + 1\right) - \left(\left(\hat{D}+1\right)(m-1)\right. \tag{20}$$

$$\hat{D} := \left(\frac{2\pi q}{PK}\right)^2 \frac{D}{\Omega}. \tag{21}$$

- Stability thresholds are obtained by setting $\kappa = 0$.
- **Theorem.** For (K, q)-ring:
  - Suppose that $\frac{K}{q} < \sqrt{2}$. Then ring is stable provided that

$$0 < D < D_{\max} := \frac{1}{\left(\frac{q}{K}\right)^2 - 1/2} \Omega \left(\frac{P}{2\pi}\right)^2 ; \tag{22}$$

    as D is increased past $D_{\max}$, the first unstable mode that is triggered corresponds to $m = 1$, $M = 1$.
  - Suppose that $\frac{K}{q} > \sqrt{2}$. Then ring is stable provided that D is sufficiently large.
    * If $K < 7$, then the ring is stable for all $D > 0$.
    * If $K > 7$ then there exists $D_{\min} > 0$ such such that the ring is stable provided that $D > D_{\min}$. It is given as the positive root of

$$\left(25\hat{D}_{\min} + 1\right)\left(9\hat{D}_{\min} + 1\right) - \left(3\left(\hat{D}_{\min} + 1\right)\left(1 - \frac{4}{K-1}\right)\right)^2,$$

      where $\hat{D}_{\min} = \left(\frac{2\pi q}{PK}\right)^2 \frac{D_{\min}}{\Omega}$. In the limit $K \to \infty$, $\hat{D}_{\min} = \left(2\sqrt{7} - 1\right)/27 \approx 0.1589$. As D is decreased below $D_{\min}$, the instability that is triggered corresponds to $m = 4$, $M = 0$.

Example: linked rings (K=2,q=2)

- Take $K = 2, q = 2$, $D = 1/2$, initial conditions $X_k(z, 0) = r_0 e^{i2\pi k/K}$. Then:

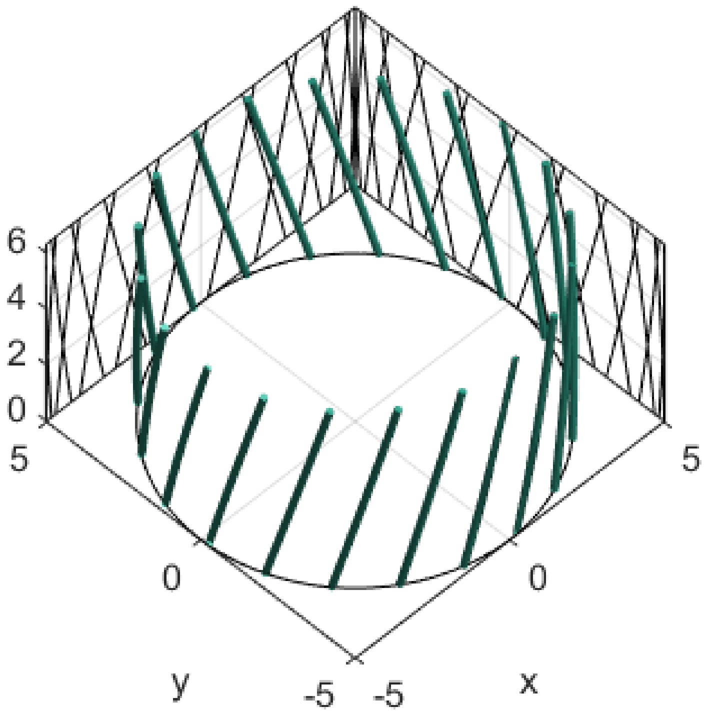
$$\Omega = \frac{1}{2r_0^2} - \frac{1}{2};$$

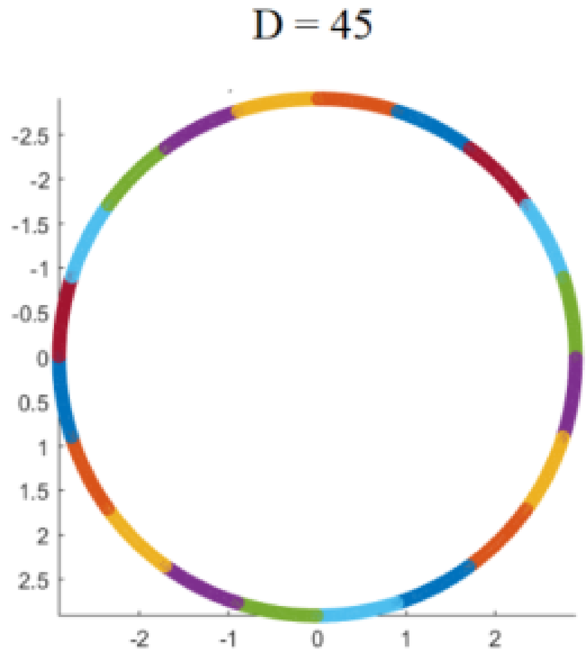
  - Stable if $r_0 < \sqrt{3}/2 = 0.866$; unstable otherwise.

  - Movies: stable ($r_0 = 0.6$), unstable ($r_0 = 1$), unstable ($r_0 = 2$),

Example: K=20, $q = 1.$

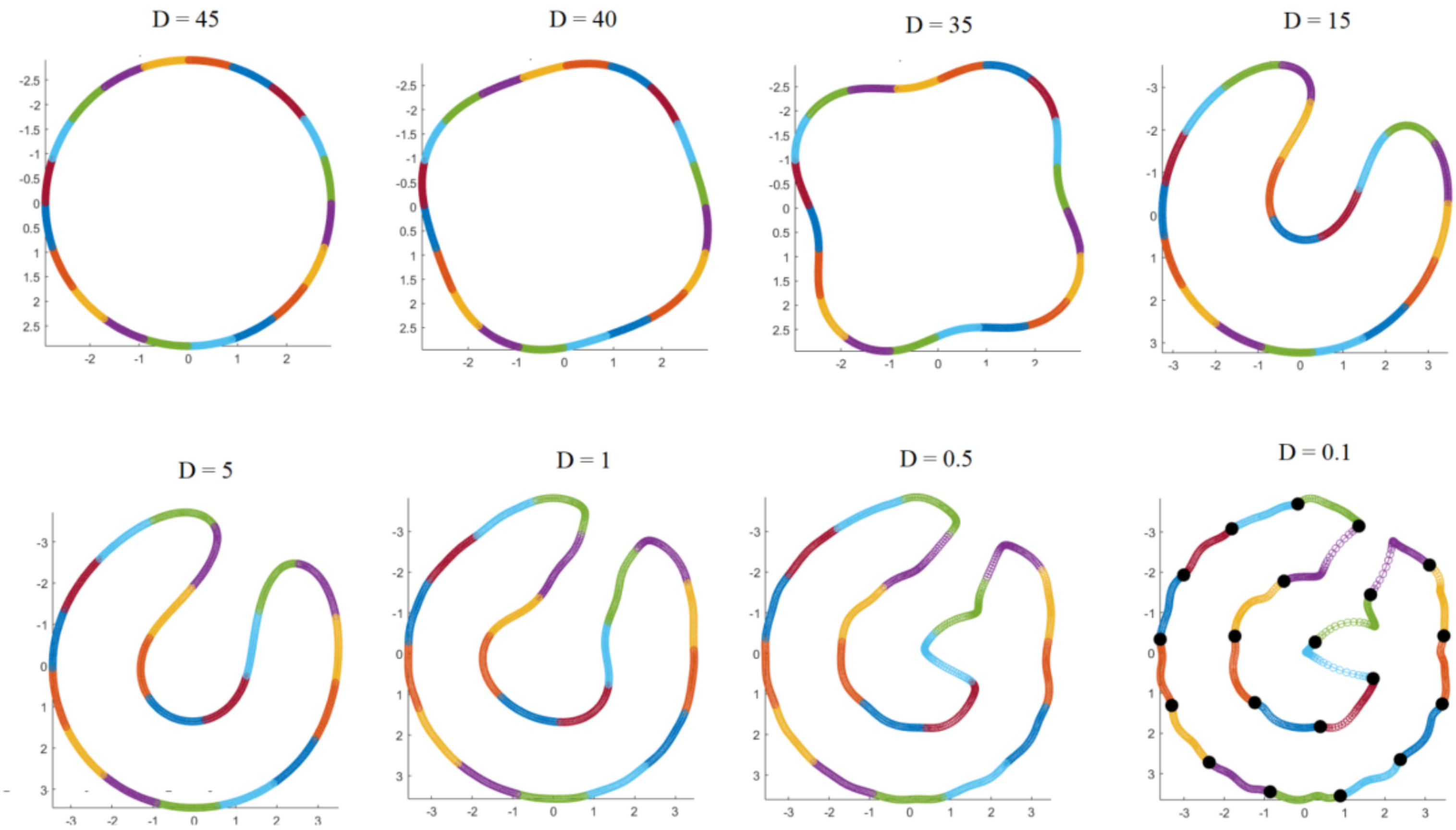
- Stable if $D > D_{\max} = 40.818$;





- Instability triggered as D is dcreased below 40.818. Supercritical bifurcation is observed. First instability corresponds to mode 4 $\longrightarrow$ “square” pattern

- For small D each slice looks like a vortex lattice... So multitude of transitions:



Twisted vortex lattices

Co-rotation frame:

$$\frac{\partial}{\partial t}\xi_k = D\frac{\partial^2}{\partial z^2}\xi_k - \Omega\xi_k + \sum_{j\neq k}\frac{\xi_k - \xi_j}{\left|\xi_k - \xi_j\right|^2}. \tag{23}$$

Point vortex lattice in 2D:

$$\xi_k(t,z) = e^{zi\omega}\hat{\xi}_k \tag{24}$$

with $\hat{\xi}_k$ satisfiying

$$0 = -\hat{\Omega}\hat{\xi}_k + \sum_{j\neq k}\frac{\hat{\xi}_k - \hat{\xi}_j}{\left|\hat{\xi}_k - \hat{\xi}_j\right|^2} \quad \text{where } \hat{\Omega} = D\omega^2 + \Omega. \tag{25}$$

Linearize:

$$\xi_k(t,z) = \xi_k + \phi_k(t), \quad \phi_k \ll 1 \tag{26}$$

then in vector form,

$$\frac{\partial}{\partial t}\phi = \left(-D\partial_{zz} - \Omega\right)\phi + e^{2i\omega z}L\bar{\phi} \tag{27}$$

where $\phi = (\phi_1, \ldots \phi_K)^T$ and $L\phi = \sum_{j\neq k}\frac{-1}{\left(\bar{\hat{\xi}}_k - \bar{\hat{\xi}}_j\right)^2}\left(\phi_k - \phi_j\right)$.

Fourier decomposition along z :

$$\phi = Ae^{i(\omega+m)z}e^{\lambda t} + Be^{i(\omega-m)z}e^{\bar{\lambda}t}.$$

yields:

$$\left(\lambda + D\left(\omega+m\right)^2 + \Omega\right)A = L\bar{B}$$
$$\left(\lambda + D\left(\omega-m\right)^2 + \Omega\right)\bar{B} = \bar{L}A$$

or

$$\left(\lambda + D\left(\omega+m\right)^2 + \Omega\right)\left(\lambda + D\left(\omega-m\right)^2 + \Omega\right) = \mu \tag{28}$$

where

$$\mu \text{ is eigenvalue of } \bar{L}L. \tag{29}$$

- The mode $m = 0$ has a zero eigenvalue $\lambda = 0$ corresponding to translation invariance of the underlying vortex lattice. This yields an eigenvalue $\mu = \left(D\omega^2 + \Omega\right)^2$ of $\bar{L}L$. Plugging back into (xxx) yields

$$0 = \lambda^2 + \left(2Dm^2 + 2D\omega^2 + 2\Omega\right)\lambda + Dm^2\left(Dm^2 - 2D\omega^2 + 2\Omega\right).$$

- **Theorem.** Consider any co-rotating configuration of vortices (vortex lattice):

$$\hat{\Omega}\hat{\xi}_k = \sum_{j\neq k} \frac{\hat{\xi}_k - \hat{\xi}_j}{\left|\hat{\xi}_k - \hat{\xi}_j\right|^2} \quad \text{where } \hat{\Omega} = Dn^2 + \Omega. \tag{30}$$

corresponding to $n-$twisted filament lattice on domain $z \in [0, 2\pi]$:

$$X_k(t, z) = e^{-\Omega t} e^{zin} \hat{\xi}_k \tag{31}$$

It is stable provided that $D < D_c$, where

$$D_c = \frac{\hat{\Omega}}{2n^2 - \frac{1}{2}} \tag{32}$$

The threshold corresponds to the instability mode $m = 1$.

- **Corrollary: Straight filaments are stable** if each cross-section is a stable vortex lattice. [No twist $\implies$ no fun].

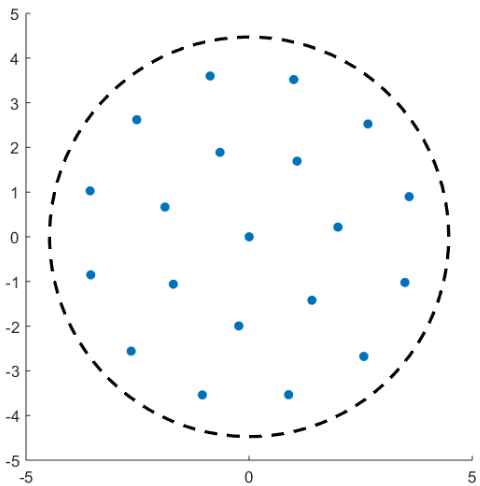
- A vortex lattice of rotation rate $\hat{\Omega}$ has radius given by

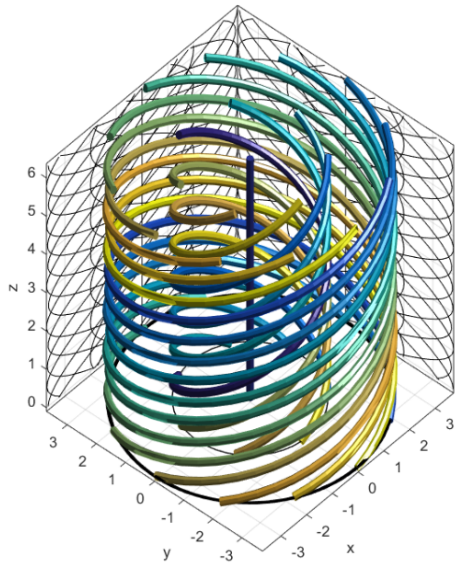
$$r^2 \sim \frac{K}{\hat{\Omega}}. \tag{33}$$

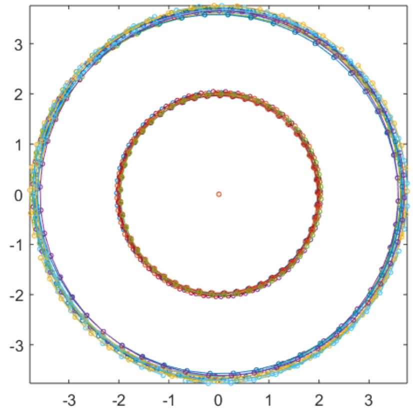
Example:

- Take $K = 20, \hat{\Omega} = 1$. Then $D_c = 0.666$. and lattice has radius size $r \sim \sqrt{20} \approx 4.72$.

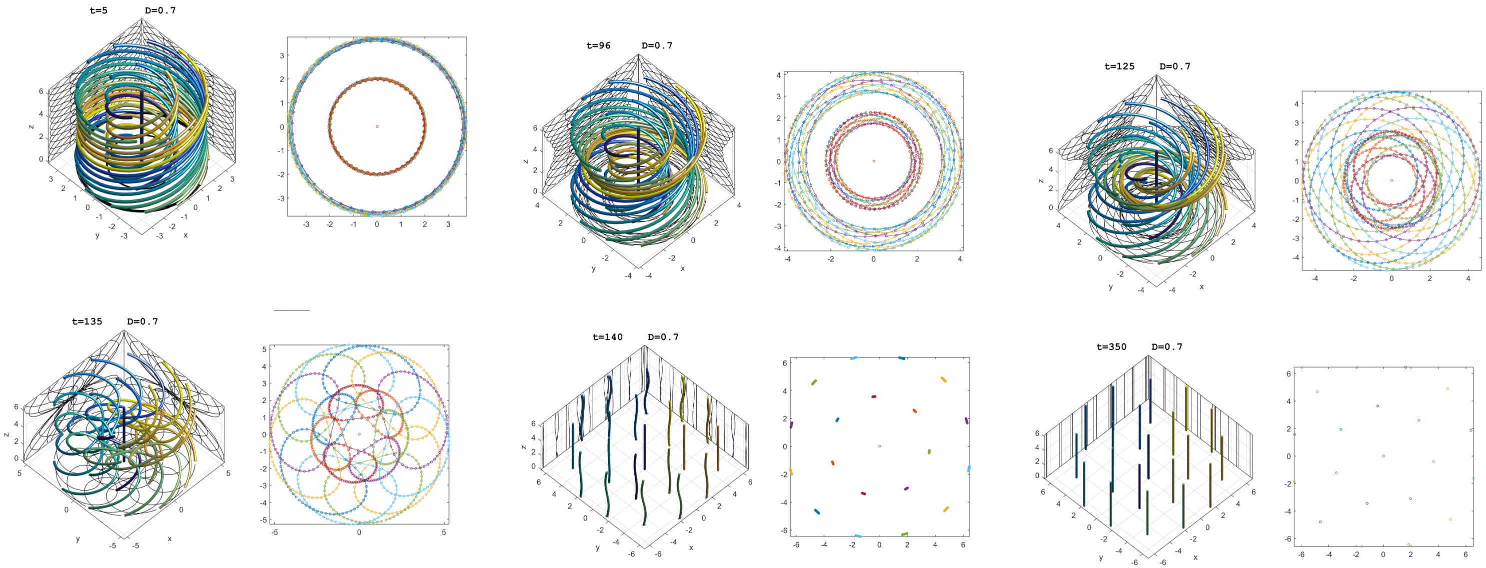
- Stable if $D = 0.6$:







- Unstable if $D = 0.7$:



Future work