Instruction

- **You are allowed 3 hours for this exam. Please pace yourself accordingly.**
- **You may not use any calculator, notes, or other assistance on this exam.**
- **In order to receive full credit, you must show your work and carefully justify your answers. The correct answer without any work will receive little or no credit.**
- **Please write neatly. Illegible answers will be assumed to be incorrect.**
- **Read the question carefully, if the method is specified you can not use other methods.**

Method of variation method:

$$y_p = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)} + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}$$

Trigonometric identity you might need for the question

$$\cos(A)\cos(B) = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$$

$$\sin(A)\sin(B) = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$

$$\sin(A)\cos(B) = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$$

Method of variation method for First order DE system:

$$\mathbf{x}(t) = e^{A(t-a)}x_a + e^{At}\int_a^t e^{-As}f(s)ds$$

where $\mathbf{x}_a = \mathbf{x}(a)$

1. **(16 points) Find the general solution of the differential equation.**
   (a) $2xyy' = x^2 + 2y^2$
   (b) $y^2y' + 2xy^3 = 6x$

2. **(10 points) Find a particular solution of the differential equation.**

$$y'' + 2y' + 5y = \sin(x)$$

3. **(10 points) Use Laplace transforms to solve the initial value problem:**

$$x'' - 6x' + 8x = 2; \quad x(0) = x'(0) = 0$$

4. (a) **(10 points) Transform the given differential equation into a equivalent system of first-order differential equations and write it in a form of $\frac{d\mathbf{y}}{dt} = A\mathbf{y} + \mathbf{f}(t)$:**

$$x^{(4)} + 6x'' - 3x' + x = \sin(3t)$$

   (b) **(10 points) Eliminate y and find the second order differential equation that x satisfies.**

$$\begin{cases} (D+2)x + (D+2)y = t \\ (2D+3)x + (D+3)y = t^2 \end{cases} \tag{1}$$

5. (a) **(10 points) Find a fundamental matrix of the given system and apply fundamental matrix solution to find a solution satisfying the initial condition.**

$$x_1' = 2x_1 - 5x_2 \quad x_2' = 4x_1 - 2x_2; \quad x_1(0) = 2, \quad x_2(0) = 3$$

   (b) **(5 points) Use the fundamental matrix you get in the (a) to compute e^{At}. where $A = \begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix}$.**

6. **(10 points) Use the Wronskian to determine whether the following three vector are independent or not.**

$$\mathbf{x_1}(t) = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}, \quad \mathbf{x_2}(t) = \begin{bmatrix} 2e^{3t} \\ 0 \\ -e^{3t} \end{bmatrix}, \quad \mathbf{x_3}(t) = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}$$

7. **(24 points) Find the general solution of the system:**
   (a)

$$\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix} \mathbf{x} \tag{2}$$

   (b)

$$\mathbf{x}' = \begin{bmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{bmatrix} \mathbf{x} \tag{3}$$

8. **(a)** **(5 points) Show that the matrix A is nilpotent and compute the matrix exponential** e^{At}

$$A = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix} \tag{4}$$

(b) **(10 points) Use the method of variation to solve the initial value problem** $\mathbf{x}' = Ax + \mathbf{f}(t)$ **where** $A = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}$ $\mathbf{f}(t) = \begin{bmatrix} 0 \\ t^{-2} \end{bmatrix}$ $x(1) = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

The rest questions are for those who need a different scheme.

9. **Find the general solution of the differential equation.**

(a) $(x+y)y' = x - y$

(b) $y'' = 2y(y')^3$

10. **Find the solution of the initial value problem**

$$y'' + 4y' + 3y = e^{-t}, y(0) = 0, y'(0) = \frac{5}{2}$$

11. **Transform the given differential equation to find a nontrivial solution such that** $x(0) = 0$**. (Hint: you can find** $x'(0)$ **by plug** $t = 0$ **into the equation)**

$$tx'' - (4t+1)x' + 2(2t+1)x = 0$$