1. (a) $2xyy' = x^2+2y^2$

$y' = \frac{1}{2}\frac{x}{y} + \frac{y}{x}$

$v = \frac{y}{x}$

$y = xv$

$y' = (xv)' = v + xv'$

$v + xv' = \frac{1}{2}\frac{1}{v} + v$

$xv' = \frac{1}{2}\frac{1}{v}$

$x\frac{dv}{dx} = \frac{1}{2v}$

$\int 2v\,dv = \int \frac{1}{x}dx$

$v^2 = \ln x + C$

$\frac{y^2}{x^2} = \ln x + C$

$y^2 = x^2(\ln x + C)$

+8

(b) $y^2y' + 2xy^3 = 6x$

$y' + 2xy = 6xy^{-2}$

$v = y^{1-(-2)} = y^3$

$v' = 3y^2\frac{dy}{dx}$

$\frac{1}{3}v' + 2xv = 6x$

$v' + 6xv = 18x$

$e^{\int 6x\,dx} = e^{3x^2}$

$e^{3x^2}v' + 6xve^{3x^2} = 18x\cdot e^{3x^2}$

$(e^{3x^2}v)' = 18x\cdot e^{3x^2}$

$e^{3x^2}v = 3e^{3x^2} + C$

$v = 3 + Ce^{-3x^2}$

$y^3 = 3 + Ce^{-3x^2}$

$y = (3+Ce^{-3x^2})^{\frac{1}{3}}$

+8

2. $y'' + 2y' + 5y = \sin(x)$

To find y_c

characteristic equation $r^2+2r+5=0$

$r = -1 \pm 2i$

$y_c = e^{-t}(\cos 2t + \sin 2t)$

$$y_p = a\cos x + b\sin x$$
$$y_p' = -a\sin x + b\cos x$$
$$y_p'' = -a\cos x - b\sin x$$
$$y_p'' + 2y_p' + 5y_p$$
$$= -a\cos x - b\sin x - 2a\sin x + 2b\cos x + 5a\cos x + 5b\sin x$$
$$= (4a+2b)\cos x + (4b-2a)\sin x$$
$$\therefore \begin{cases} 4a+2b=0 \\ 4b-2a=1 \end{cases} \Rightarrow \begin{cases} a=-\frac{1}{10} \\ b=\frac{1}{5} \end{cases}$$

+10

$$\therefore y_p = -\frac{1}{10}\cos x + \frac{1}{5}\sin x$$

3. $x'' - 6x' + 8x = 2 \quad x(0) = x'(0) = 0$

$$\mathcal{L}\{x''\} - 6\mathcal{L}\{x'\} + 8\mathcal{L}\{x\} = \mathcal{L}\{2\}$$
$$s^2\mathcal{L}\{x\} - 6s\mathcal{L}\{x\} + 8\mathcal{L}\{x\} = \frac{2}{s}$$
$$(s^2-6s+8)\mathcal{L}\{x\} = \frac{2}{s}$$
$$\mathcal{L}\{x\} = \frac{2}{s(s^2-6s+8)}$$
$$\mathcal{L}\{x\} = \frac{2}{s[(s-3)^2-1]}$$
$$x(t) = \int_0^t 2\cdot e^{3v}\sinh v\,dv$$
$$= \int_0^t e^{3v}(e^v - e^{-v})\,dv$$
$$= \int_0^t (e^{4v} - e^{2v})\,dv$$
$$= \left[\frac{1}{4}e^{4v} - \frac{1}{2}e^{2v}\right]_0^t$$
$$= \frac{1}{4}e^{4t} - \frac{1}{2}e^{2t} + \frac{1}{4}$$

+10

4. (a) define $y_0 = x$

$$y_1 = x' = y_0'$$
$$y_2 = x'' = y_1'$$
$$y_3 = x''' = y_2'$$
$$y_4 = x^{(4)} = y_3'$$

+10

$$\therefore \begin{cases} y_0' = y_1 \\ y_1' = y_2 \\ y_2' = y_3 \\ y_3' = -y_0 + 3y_1 - 6y_2 + \sin(3t) \end{cases} \Rightarrow \begin{bmatrix} y_0' \\ y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 3 & -6 & 0 \end{bmatrix} \vec{y} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin(3t) \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 3 & -6 & 0 \end{bmatrix} \quad f(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin(3t) \end{bmatrix}$$

(b). $\begin{cases} (D+2)x + (D+2)y = t \\ (2D+3)x + (D+3)y = t^2 \end{cases}$

$L = \begin{bmatrix} D+2 & D+2 \\ 2D+3 & D+3 \end{bmatrix}$

$|L| = (D+2)(D+3) - (D+2)(2D+3)$

$= -(D+2)D \neq 0$

use cramer rules.

$|L|x = \begin{vmatrix} t & D+2 \\ t^2 & D+3 \end{vmatrix}$

$x = \dfrac{t(D+3) - t^2(D+2)}{-D(D+2)}$

$= \dfrac{-2t^2+t+1}{-D(D+2)}$

+10

$x'' + 2x' = 2t^2 - t - 1$

$|L|y = \begin{vmatrix} D+2 & t \\ 2D+3 & t^2 \end{vmatrix}$

$y = \dfrac{t^2(D+2) - t(2D+3)}{-D(D+2)}$

$= \dfrac{2t^2-t-2}{-D(D+2)}$

$\therefore y'' + 2y' = -2t^2 + t + 2$

5. (a) $\begin{cases} x_1' = 2x_1 - 5x_2 \\ x_2' = 4x_1 - 2x_2 \end{cases}$ $\quad x_1(0) = 2, \quad x_2(0) = 3.$

$A = \begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix}$

$\det(A - \lambda I) = (2-\lambda)(-2-\lambda) + 20$

$= \lambda^2 + 16 = 0$

$\therefore \lambda = \pm 4i$

$(A - \lambda I)\vec{v} = 0$

$\begin{bmatrix} 2-4i & -5 \\ 4 & -2-4i \end{bmatrix} \vec{v} = 0 \qquad v_1 = \begin{bmatrix} 5 \\ 2-4i \end{bmatrix}$

$\vec{v} e^{\lambda t} = \begin{bmatrix} 5 \\ 2-4i \end{bmatrix} (\cos 4t + i \sin 4t) = \begin{bmatrix} 5\cos 4t + 5i \sin 4t \\ 2\cos 4t + 2i \sin 4t - 4i \cos 4t + 4 \sin 4t \end{bmatrix}$

$$\Phi(t) = \begin{bmatrix} 5\cos4t & 5\sin4t \\ 2\cos4t + 4\sin4t & 2\sin4t - 4\cos4t \end{bmatrix}$$

$$\Phi(0) = \begin{bmatrix} 5 & 0 \\ 2 & -4 \end{bmatrix}$$

$$\Phi^{-1}(0) = \frac{1}{-20} \begin{bmatrix} -4 & 0 \\ -2 & 5 \end{bmatrix}$$

$$X(t) = \Phi(t) \cdot \Phi^{-1}(0) \cdot X(0)$$

+10

$$= \begin{bmatrix} 5\cos4t & 5\sin4t \\ 2\cos4t + 4\sin4t & 2\sin4t - 4\cos4t \end{bmatrix} \cdot -\frac{1}{20} \begin{bmatrix} -4 & 0 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \cos4t + \frac{1}{2}\sin4t & -\frac{5}{4}\sin4t \\ \sin4t & \cos4t - \frac{1}{2}\sin4t \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cos4t - \frac{11}{4}\sin4t \\ 3\cos4t + \frac{1}{2}\sin4t \end{bmatrix}$$

(b). $e^{At} = \Phi(t) \cdot \Phi^{-1}(0)$

$$= \begin{bmatrix} 5\cos4t & 5\sin4t \\ 2\cos4t + 4\sin4t & 2\sin4t - 4\cos4t \end{bmatrix} \begin{bmatrix} \frac{1}{5} & 0 \\ \frac{1}{10} & -\frac{1}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \cos4t + \frac{1}{2}\sin4t & -\frac{5}{4}\sin4t \\ \sin4t & \cos4t - \frac{1}{2}\sin4t \end{bmatrix}$$

+5

6. $$W = \begin{vmatrix} 2e^{t} & 2e^{3t} & 2e^{5t} \\ 2e^{t} & 0 & -2e^{5t} \\ e^{t} & -e^{3t} & e^{5t} \end{vmatrix}$$

$$= -2e^{t}(2e^{8t} + 2e^{8t}) + 2e^{5t}(-2e^{4t} - 2e^{4t})$$

$$= -8e^{9t} - 8e^{9t}$$

$$= -16e^{9t} \neq 0$$

+10

$\therefore$ $X_1(t)$ $X_2(t)$ $X_3(t)$ are linear independent.

7. (a) $$X' = \begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix} X$$

$$A = \begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)(9-\lambda) + 16 = (\lambda - 5)^2 = 0$$

$(A-5I)\vec{v}=0$

$\begin{bmatrix}-4 & -4\\ 4 & 4\end{bmatrix}\vec{v}=0.$ $\quad v_1=\begin{bmatrix}1\\-1\end{bmatrix}$

$(A-\lambda I)^2 v_2=0$

since $(A-\lambda I)^2=0$ v_2 can be any vector

choose $v_2=\begin{bmatrix}1\\0\end{bmatrix}$

$v_1=(A-\lambda I)v_2$

$=\begin{bmatrix}-4\\4\end{bmatrix}$

+12

$\vec{x}(t)=C_1\begin{bmatrix}-4\\4\end{bmatrix}e^{5t}+C_2\begin{bmatrix}-4t+1\\4t\end{bmatrix}e^{5t}$

$x_1(t)=[4C_1+(-4t+1)C_2]e^{5t}$

$x_2(t)=(4C_1+4tC_2)e^{5t}$

(b). $A=\begin{bmatrix}-2 & -9 & 0\\ 1 & 4 & 0\\ 1 & 3 & 1\end{bmatrix}$

$\det(A-\lambda I)=(1-\lambda)[(-2-\lambda)(4-\lambda)+9]=(1-\lambda)^3$

$(A-I)\vec{v}=0$

$\begin{bmatrix}-3 & -9 & 0\\ 1 & 3 & 0\\ 1 & 3 & 0\end{bmatrix}\vec{v}=0.$ $\quad \vec{v}=b\begin{bmatrix}3\\-1\\0\end{bmatrix}+c\begin{bmatrix}0\\0\\1\end{bmatrix}$

$(A-\lambda I)^2 V_2=0$ $\quad\because (A-\lambda I)^2=0$

$\therefore$ choose $V_2=\begin{bmatrix}1\\0\\0\end{bmatrix}$

+12

$V_1=(A-\lambda I)V_2=\begin{bmatrix}-3\\1\\1\end{bmatrix}$

$V_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$

$\therefore x(t)=C_1\begin{bmatrix}-3\\1\\1\end{bmatrix}e^{t}+C_2\begin{bmatrix}-3t+1\\t\\t\end{bmatrix}e^{t}+C_3\begin{bmatrix}0\\0\\1\end{bmatrix}e^{t}$

$\therefore\ x_1(t) = [-3c_1 + (-3t+1)c_2]e^t$

$x_2(t) = (c_1 + c_2 t)e^t$

$x_3(t) = (c_1 + c_2 t + c_3)e^t$

8\. (a) $A = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}$

$A^2 = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore A^n = 0$ if $n \geq 1$.

+5

$\therefore$ A is nilpotent

$\therefore e^{At} = I + At$

$= \begin{bmatrix} 3t+1 & -t \\ 9t & -3t+1 \end{bmatrix}$

(b) $e^{At} = \begin{bmatrix} 3t+1 & -t \\ 9t & -3t+1 \end{bmatrix}$

$\therefore e^{-As} f(s) = \begin{bmatrix} 1-3s & s \\ -9s & 1+3s \end{bmatrix}\begin{bmatrix} 0 \\ s^{-2} \end{bmatrix}$

$= \begin{bmatrix} s^{-1} \\ s^{-2} + 3s^{-1} \end{bmatrix}$

$\int_1^t e^{-As} f(s)\,ds = \begin{bmatrix} \ln t \\ 3\ln t - \frac{1}{t} + 1 \end{bmatrix}$

+9

$X(t) = e^{A(t-1)} X(1) + e^{At}\int_1^t e^{-As} f(s)\,ds$

$= \begin{bmatrix} 1+3(t-1) & -(t-1) \\ 9(t-1) & 1-3(t-1) \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix} + \begin{bmatrix} 1+3t & -t \\ 9t & 1-3t \end{bmatrix}\begin{bmatrix} \ln t \\ 3\ln t - \frac{1}{t} + 1 \end{bmatrix}$

$= \begin{bmatrix} 2t+1 \\ 6t+1 \end{bmatrix} + \begin{bmatrix} \ln t + 1 - t \\ 3\ln t - 3t + 4 - \frac{1}{t} \end{bmatrix}$

$= \begin{bmatrix} \ln t + t + 2 \\ 3\ln t + 3t - \frac{1}{t} + 5 \end{bmatrix}$