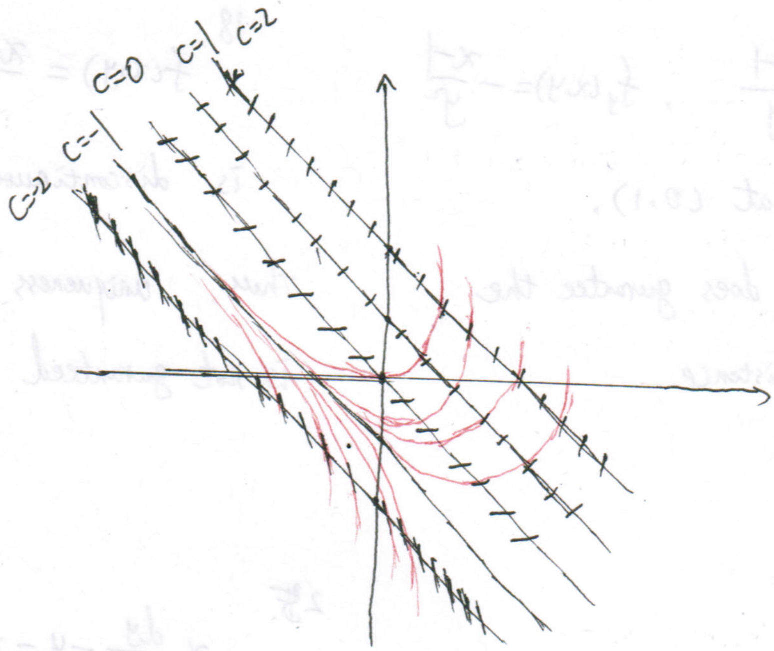
Part I.

1.

isocline:

$x+y=C$

$y=-x+C$



2.

a) $$\frac{dP}{dt} = P(100-P)$$

see last page for 3.

separating variable: $$\frac{dP}{P(100-P)} = dt$$

Hence $$\int \frac{dP}{P(100-P)} = \int dt$$

$$\frac{1}{100}\int\left(\frac{1}{P}+\frac{1}{100-P}\right)dP = \int dt$$

$$\frac{1}{100}\left(\ln P - \ln(100-P)\right) = t+C$$

$$\frac{P}{100-P} = e^{100(t+C)} \Rightarrow P = \frac{100}{1+e^{-100(t+C)}}$$

b) $\lim\limits_{t\to+\infty} P = 100$

Part II.

§1.3.

17:

$$f(x,y) = \frac{x-1}{y}, \quad f_y(x,y) = -\frac{x-1}{y^2}$$

is continuous at $(0,1)$.

Thus Theorem 1 does gurantee the uniqueness and existence.

18,

$$f(x,y) = \frac{x-1}{y}$$

is discontinuous at $(1,0)$

Thus, uniqueness and existence is not guranteed.

§1.4

23

$$\frac{dy}{dx} + 1 = 2y$$

$$\frac{dy}{dx} = 2y - 1$$

$$\frac{dy}{2y-1} = dx$$

$$\frac{1}{2}\ln(2y-1) = x + C$$

$$2y - 1 = e^{2x} \cdot B \quad (B = e^{2C})$$

$$y = \frac{Be^{2x}+1}{2}$$

~~23~~ $y(1) = 1 \Rightarrow \frac{Be^2+1}{2} = 1$

$$\Rightarrow B = e^{-2}$$

$$y = \frac{e^{2x-2}+1}{2}$$

25.

$$x\frac{dy}{dx} - y = 2x^2 y$$

$$\frac{dy}{dx} = \frac{(2x^2+1)y}{x}$$

$$\frac{dy}{y} = \frac{2x^2+1}{x}dx$$

$$\ln y = x^2 + \ln x + C$$

$$y(1) = 1$$

$$\Rightarrow \ln 1 = 1^2 + 0 + C$$

$$\Rightarrow C = -1$$

$$\ln y = x^2 + \ln x - 1$$

$$\Leftrightarrow \quad y = xe^{x^2-1}$$

S 1.5

4. $y' - 2xy = e^{x^2}$ $\cdots ①$

integrating factor $e^{\int -2x\,dx} = e^{-x^2}$

Multiply ① by e^{-x^2}:

$$e^{-x^2}y' - 2xe^{-x^2}y = 1$$

$$\Longleftrightarrow (e^{-x^2}y)' = 1$$

$$\Longleftrightarrow e^{-x^2}y = x + C$$

$$\Longleftrightarrow y = (x+c)e^{x^2}$$

8. $3xy' + y = 12x$

$\overbrace{\text{Standard}}^{\text{linear}}$ form $y' + \frac{y}{3x} = 4 \cdots ①$

integrating factor $e^{\int \frac{1}{3x}dx} = x^{\frac{1}{3}}$

Multiply ① by $x^{\frac{1}{3}}$:

$$x^{\frac{1}{3}}y' + \frac{1}{3}x^{-\frac{2}{3}}y = 4x^{\frac{1}{3}}$$

$$\Longleftrightarrow (x^{\frac{1}{3}}y)' = 4x^{\frac{1}{3}}$$

$$\Longleftrightarrow x^{\frac{1}{3}}y = 3x^{\frac{4}{3}} + C$$

$$\Longleftrightarrow y = 3x + Cx^{-\frac{1}{3}}$$

S 1.6

1. $(x+y)y' = x - y$

$$y' = \frac{x-y}{x+y}$$

$$y' = \frac{1-\frac{y}{x}}{1+\frac{y}{x}}$$

$$v = \frac{y}{x} \Rightarrow xv = y \Rightarrow xv' + v = y'$$

$$xv' + v = \frac{1-v}{1+v}$$

$$v' = \frac{1-2v-v^2}{1+v}\cdot\frac{1}{x}$$

$$\frac{1+v}{1-2v-v^2}\,dv = \frac{1}{x}dx$$

$$\frac{1+v}{2-(1+v)^2}\,dv = \frac{1}{x}dx$$

$$-\frac{1}{2}\ln\left[(1+v)^2-2\right] = \ln x + C$$

$$(1+v)^2 - 2 = \frac{B}{x^2} \qquad B = e^{-2C}$$

$$\Longleftrightarrow (1+\frac{y}{x})^2 - 2 = \frac{B}{x^2}$$

$$\Longleftrightarrow y^2 + 2xy - x^2 = B.$$

See next page for method 2.

another way to solve S1.6 1

$(x+y)y' = x-y$

$\Longleftrightarrow (x-y)\,dx-(x+y)\,dy=0$

$M(x,y) = x-y$

$N(x,y) = -(x+y)$

$\left.\begin{aligned}\frac{\partial M}{\partial y} &= -1\\ \frac{\partial N}{\partial x} &= -1\end{aligned}\right] \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -1.$

$\Rightarrow$ It's a exact DE.

$\frac{\partial F}{\partial x} = M = x-y$

integrating with respect to x

$F = \frac{x^2}{2} - yx + g(y)$

then $\frac{\partial F}{\partial y} = g'(y) - x$

since $\frac{\partial F}{\partial y} = N = -(x+y)$.

Thus $g'(y) - x = -(x+y) \Rightarrow g'(y) = -y$

$\Rightarrow g(y) = -\frac{y^2}{2}$

$\Rightarrow F = \frac{x^2}{2} - yx - \frac{y^2}{2}$

Hence the general solution is $F = C \Longleftrightarrow \frac{x^2}{2} - yx - \frac{y^2}{2} = C$.

S1.6

9. $x^2y' = xy + y^2$

$y' = \frac{y}{x} + \left(\frac{y}{x}\right)^2$

$v = \frac{y}{x}, \quad v + xv' = y'$

$v + xv' = v + v^2$

$v' = \frac{v^2}{x}$

$\frac{dv}{v^2} = \frac{dx}{x}$

$-\frac{1}{v} = \ln x + C$

$v = -\frac{1}{\ln x + C} \Rightarrow y = -\frac{x}{\ln x + C}$

22. $x^2y' + 2xy = 5y^4$

$y' + \frac{2}{x}y = 5\frac{y^4}{x^2}$ --- ①

It is Bernoulli equation, $n=4$

$v = y^{1-n} = y^{-3}, \quad v' = -3y^{-4}y'$

Rewrite ① as (divide both side by y^4)

$y^{-4}y' + \frac{2}{x}y^{-3} = \frac{5}{x^2}$

Thus

$-\frac{v'}{3} + \frac{2}{x}v = \frac{5}{x^2}$

using integrating factor to solve v

we get $v = \frac{15}{7x} + Cx^6$

Hence $y^{-3} = \frac{15}{7x} + Cx^6$

§1.6

23. $xy' + 6y = 3xy^{4/3}$

$$y' + \frac{6}{x} y = 3y^{4/3} \quad \cdots ①$$

It is Bernoulli equation, $n = \frac{4}{3}$

Set $v = y^{1-n} = y^{-\frac{1}{3}}$, $v' = -\frac{1}{3} y^{-\frac{4}{3}} y'$

Rewrite ① (divide both side by $y^{4/3}$) as.

$$y^{-\frac{4}{3}} y' + \frac{6}{x} y^{-\frac{1}{3}} = 3$$

Thus

$$-3v' + \frac{6}{x} v = 3$$

using integrating factor to solve it

We obtain:

$$v = x + Cx^2.$$

Thus $$y^{-\frac{1}{3}} = x + Cx^2.$$

43. $xy'' = y'$

Let $v = y'$

then $y'' = v'$

$$xv' = v$$

$$\frac{dv}{dx} = \frac{v}{x}$$

$(B_1 = e^C)$

$$\frac{dv}{v} = \frac{dx}{x} \Rightarrow \ln v = \ln x + C \Rightarrow v = \cancel{B_1 e^x} B_1 x$$

Hence $y' = \cancel{B_1 e^x}\ B_1 x \Rightarrow y = \cancel{B_1 e^x}\ \frac{1}{2} B_1 x^2 + B_2.$

31. $(2x+3y)\,dx + (3x+2y)\,dy = 0$

$$M = 2x + 3y$$
$$N = 3x + 2y$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 3 \\ \frac{\partial N}{\partial x} &= 3 \end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Hence it is exact.

Then $$\frac{\partial F}{\partial x} = M = 2x + 3y$$

$$\Rightarrow F = x^2 + 3xy + g(y)$$

$$\left.\begin{aligned} \frac{\partial F}{\partial y} &= 3x + g'(y) \\ \text{since } \frac{\partial F}{\partial y} &= N = 3x + 2y \end{aligned}\right\} \Rightarrow 3x + g'(y) = 3x + 2y$$

$$\Rightarrow g'(y) = 2y \Rightarrow g(y) = y^2$$

$$\Rightarrow F = x^2 + 3xy + y^2$$

Therefore the general solution is

$$x^2 + 3xy + y^2 = C.$$

53. $y'' = 2yy'$

Let $v = y'$

$$y'' = \frac{dv}{dx} = \frac{dv}{dy} \cdot \frac{dy}{dx} = \frac{dv}{dy} \cdot v$$

Plug back $$\frac{dv}{dy} \cdot v = 2y \cdot v$$

$$\frac{dv}{dy} = 2y \Rightarrow v = y^2 + C_1 \Rightarrow y' = y^2 + C_1$$

$$\frac{dy}{y^2 + C_1} = dx \Rightarrow \frac{1}{\sqrt{C_1}} \arctan\left(\frac{y}{\sqrt{C_1}}\right) = x + C_2.$$

§ 2.1

33. $y''-3y'+2y=0$

characteristic equation

$$r^2-3r+2=0$$

Solve it: $r=1$, $r=2$.

Hence the general solution is

$$y=C_1e^{t}+C_2e^{2t}.$$

40. $9y''-12y'+4y=0$

Charateristic equation

$$9r^2-12r+4=0$$

$$(3r-2)^2=0$$

only one root $r=\frac{2}{3}$

Hence the general solution is

$$y=(C_1+C_2t)e^{\frac{2}{3}t}.$$

S. 2.2

8

Their Wronskian is

$$W=\begin{vmatrix} e^{x} & e^{2x} & e^{3x} \\ e^{x} & 2e^{2x} & 3e^{3x} \\ e^{x} & 4e^{2x} & 9e^{3x}\end{vmatrix}$$

$$=e^{x}\begin{vmatrix} 2e^{2x} & 3e^{3x} \\ 4e^{2x} & 9e^{3x}\end{vmatrix}-e^{x}\begin{vmatrix} e^{2x} & e^{3x} \\ 4e^{2x} & 9e^{3x}\end{vmatrix}+e^{x}\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x}\end{vmatrix}$$

$$=e^{x}\cdot 6e^{5x}-e^{x}\cdot 5e^{5x}+e^{x}\cdot e^{5x}$$

$$=2e^{6x}\neq 0$$

Because $W\neq 0$ everywhere, it follows that f, g and h are linely independent.

21.

the general solution

$$y=y_c+y_p$$

$$=C_1\cos x+C_2\sin x+3x$$

$$y'=-C_1\sin x+C_2\cos x+3$$

$y(0)=2$, $y'(0)=-2$

$$\begin{cases} C_1\cos 0+C_2\sin 0+3\cdot 0=2 \\ -C_1\sin 0+C_2\cos\cdot 0+3=-2\end{cases}$$

$$\Rightarrow \begin{cases} C_1=2 \\ C_2=-5\end{cases}$$

Hence

$$y=2\cos x-5\sin x+3x$$

S2.3

11. $y^{(4)} - 8y^{(3)} + 16y'' = 0$

Characteristic equation

$$r^4 - 8r^3 + 16r^2 = 0$$

$$r^2(r-4)^2 = 0$$

$r=0$ with multiplicity 2

$r=4$ with multiplicity 2

Thus the general solution is

$$y = c_1 + c_2 t + e^{4t}(c_3 + c_4 t)$$

S2.5

2. $y'' - y' - 2y = 3x + 4 \quad \cdots \textcircled{1}$

The characteristic equation

$r^2 - r - 2 = 0$ has root $r=-1, r=2$.

So $y_c = c_1 e^{-x} + c_2 e^{2x}$

We try $y_p = Ax + B$.

Substitute y_p into equation ①,

$$-A - 2(Ax+B) = 3x + 4$$

equating the coefficient $\begin{cases} -2A = 3 \\ -2B - A = 4 \end{cases}$

$$A = -\frac{3}{2}, \quad B = -\frac{5}{4}$$

Thus $y_p = -\frac{3}{2}A - \frac{5}{4}$.

24. $2y^{(3)} - 3y'' - 2y' = 0$

Characteristic equation

$$2r^3 - 3r^2 - 2r = 0$$

$$r(2r+1)(r-2) = 0$$

$$r = 0, \ -\tfrac{1}{2}, \ 2$$

Hence the general solution is

$$y = c_1 + c_2 e^{-\frac{1}{2}t} + c_3 e^{2t}$$

13. $y'' + 2y' + 5y = e^x \sin x$

The characteristic equation

$r^2 + 2r + 5$ has root $r = -1 \pm 2i$

So $y_c = \cancel{e^{-2x}(c_1\cos 2x + c_2 \sin 2x)}$
$e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$

We try $y_p = e^x(A\sin x + B\cos x)$

$$y_p' = e^x[(A-B)\sin x + (A+B)\cos x]$$

$$y_p'' = e^x[-2B\sin x + 2A\cos x]$$

Then

$$e^x(-2B\sin x + 2A\cos x + 2[(A-B)\sin x + (A+B)\cos x] + 5(A\sin x + B\cos x)] = e^x \sin x$$

$\begin{cases} 7A - 4B = 1 \\ 4A + 7B = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{7}{65} \\ B = -\frac{4}{65} \end{cases}$

Thus $y_p = \left(\frac{7}{65}\sin x - \frac{4}{65}\cos x\right)e^x$

§ 2.8

3. $y'' + \lambda y = 0$, $y(-\pi) = 0$, $y(\pi) = 0$

If $\lambda = 0$, then $y'' = 0 \Rightarrow y = Ax + B$

imposing boundary condition yields $\begin{cases} -A\pi + B = 0 \\ A\pi + B = 0 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = 0. \end{cases}$

only trivial solution is possible, thus $\lambda = 0$ is not a eigenvalue.

If $\lambda > 0$, write $\lambda = a^2$ then the general solution is

$y = A\cos ax + B\sin ax$.

The boundary condition implys $\begin{cases} A\cos a\pi + B\sin a\pi = 0 & \cdots ① \\ A \overset{\cos(-a\pi)}{\cancel{\cos a\pi}} + B\sin(-a\pi) = 0 & \cdots ② \end{cases}$

① + ② give us $2A\cos a\pi = 0 \Rightarrow A = 0$ or $\cos a\pi = 0$

i) If $A = 0$, then ~~cos aπ~~ ① gives us $B\sin a\pi = 0 \Rightarrow \sin a\pi = 0$

then $a\pi = k\pi \Rightarrow a = k$. the associated solution is $y = B\sin k x$

ii) If $\cos a\pi = 0$, then $a\pi = \frac{2k-1}{2}\pi \Rightarrow a = \frac{2k-1}{2}$,

the associated solution is $y = A\cos(\frac{2k-1}{2}x)$.

Therefore. $\lambda = k^2$ or $(\frac{2k-1}{2})^2$

the associated solution is $\sin(kx)$ and ~~or~~ $\cos(\frac{2k-1}{2}x)$ respectively.