§ 4.1

3. $f(t) = e^{3t+1}$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} \cdot e^{3t+1} dt$$

$$= \int_0^\infty e^{(-s+3)t+1} dt$$

$$= \frac{1}{-s+3} e^{(-s+3)t+1} \Big|_0^\infty$$

$$= \frac{-e}{-s+3} \qquad s > 3$$

4. $f(t) = \cos t$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} \cdot \cos t \, dt$$

$$= \frac{e^{-st}(\sin t - s\cos t)}{s^2+1} \Big|_0^\infty$$

$$= \frac{s}{s^2+1}$$

8. $f(t) = \begin{cases} 0 & 0 < t \le 1 \\ 1 & 1 < t \le 2 \\ 0 & t > 2 \end{cases}$

$$\mathcal{L}(f(t)) = \int_0^\infty f(t) e^{-st} dt$$

$$= \int_1^2 e^{-st} dt$$

$$= \frac{e^{-st}}{-s} \Big|_1^2$$

$$= \frac{e^{-2s}}{-s} - \frac{e^{-s}}{-s}$$

18. $f(t) = \sin 3t \cos 3t$

$$= \frac{1}{2} \sin 6t$$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} \cdot \frac{1}{2} \sin 6t \, dt$$

$$= \frac{1}{2} \frac{e^{-st}(s \sin(6t) + 6\cos(6t))}{s^2+6^2} \Big|_0^\infty$$

$$= \frac{3}{s^2+6^2}$$

29. $$F(s) = \frac{5-3s}{s^2+9} = \frac{5}{s^2+9} - \frac{3s}{s^2+9}$$

$$\mathcal{L}^{-1}(F(s)) = \frac{5}{3} \sin 3t - 3\cos 3t$$

30. $$F(s) = \frac{9+s}{4-s^2} = \frac{9}{4-s^2} + \frac{s}{4-s^2}$$

$$\mathcal{L}^{-1}\left(\frac{9}{4-s^2}\right) = -\frac{9}{2} \sinh 2t$$

$$\mathcal{L}^{-1}\left(\frac{s}{4-s^2}\right) = -\cosh 2t$$

$$\mathcal{L}^{-1}(F(s)) = -\frac{9}{2} \sinh 2t - \cosh 2t$$

32. $$F(s) = \frac{2e^{-3s}}{s} = 2 \cdot \frac{e^{-3s}}{s}$$

$$\mathcal{L}^{-1}(F(s)) = 2\,u(t-3)$$

§ 4.2

3. $x'' - x' - 2x = 0$, $x(0) = 0$, $x'(0) = 2$

$$\mathcal{L}(x'(t)) = s\,\mathcal{L}(x(t)) - x(0) = sX(s).$$

$$\mathcal{L}(x''(s)) = s^2\mathcal{L}(x(t)) - s\,x(0) - x'(0) = s^2X(s) - 2$$

The transform equation is

$$s^2X(s) - 2 - s\,X(s) - 2X(s) = 0$$

$$X(s) = \frac{2}{s^2 - s - 2} = \frac{2}{(s-2)(s+1)}$$

$$= \frac{2}{3}\left(\frac{1}{s-2} - \frac{1}{s+1}\right)$$

$$\mathcal{L}^{-1}(X(s)) = \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t}$$

5. $x'' + x = \sin 2t$; $x(0) = 0$, $x'(0) = 0$

$$\mathcal{L}(x'(t)) = s\,\mathcal{L}(x(t)) - x(0) = sX(s)$$

$$\mathcal{L}(x''(t)) = s^2\mathcal{L}(x(t)) - s\,x(0) - x'(0) = s^2X(s)$$

$$\mathcal{L}(\sin 2t) = \frac{2}{s^2+4}$$

The transform equation is

$$s^2X(s) + X(s) = \frac{2}{s^2+4}$$

$$X(s) = \frac{2}{(s^2+4)(s^2+1)} = \frac{2}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)$$

$$\mathcal{L}^{-1}(X(s)) = \frac{2}{3}\sin t - \frac{1}{3}\sin 2t$$

17. $$F(s) = \frac{1}{s(s-3)} = \frac{1}{3}\left(\frac{1}{s-3} - \frac{1}{s}\right)$$

$$\mathcal{L}^{-1}(F(s)) = \frac{1}{3}e^{3t} - \frac{1}{3}$$

19. $$F(s) = \frac{1}{s(s^2+4)} = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$\mathcal{L}^{-1}(F(s)) = \frac{1}{4}\left(1 - \cos(2t)\right)$$

$= \frac{1}{2}\sin^2 t$

23. $$F(s) = \frac{1}{s^2(s^2-1)} = \frac{1}{s^2-1} - \frac{1}{s^2}$$

$$\mathcal{L}^{-1}(F(s)) = \sinh t - t$$

§ 4.3

6. $$F(s) = \frac{s-1}{(s+1)^3} = \frac{1}{(s+1)^2} - \frac{2}{(s+1)^3}$$

$$\mathcal{L}^{-1}(F(s)) = \cancel{\frac{t^2e^{-t}}{2} - \frac{2t^3e^{-t}}{3!}}$$

$$te^{-t} - t^2e^{-t}$$

9. $$F(s) = \frac{3s+5}{s^2-6s+25} = \frac{3(s-3)+14}{(s-3)^2+4^2}$$

$$= \frac{3(s-3)}{(s-3)^2+4^2} + \frac{4}{(s-3)^2+4^2}\cdot\frac{14}{4}$$

$$\mathcal{L}^{-1}(F(s)) = 3e^{3t}\cos 4t + \frac{7}{2}e^{3t}\sin 4t$$

§ 4.3

11.

$$F(s) = \frac{1}{s^2-4} = \frac{1}{4}\left(\frac{1}{s-2} - \frac{1}{s+2}\right).$$

$$\mathcal{L}^{-1}(F(s)) = \frac{1}{4}\left(e^{2t} - e^{-2t}\right) = \frac{1}{2}\sinh 2t$$

27. $x'' + 6x' + 25x = 0;\quad x(0)=2,\ x'(0)=3$

$$\mathcal{L}(x') = s\,\mathcal{L}(x(t)) - x(0) = sX(s) - 2$$

$$\mathcal{L}(x'') = s^2\,\mathcal{L}(x(t)) - s\,x(0) - x'(0) = s^2X(s) - 2s - 3$$

The Transform Equation is

$$s^2X(s) - 2s - 3 + 6(sX(s) - 2) + 25X(s) = 0$$

$$X(s) = \frac{2s+15}{s^2+6s+25}$$

$$= \frac{2(s+3)+9}{(s+3)^2+16}$$

$$= \frac{2(s+3)}{(s+3)^2+16} + \frac{4}{(s+3)^2+16}\cdot\frac{9}{4}$$

$$\mathcal{L}^{-1}(X(s)) = 2e^{-3t}\cos 4t + \frac{9}{4}e^{-3t}\sin 4t$$

§ 4.4

9. use Theorem 1, P_{298}

$$F(s) = \frac{1}{(s^2+9)^2} = \frac{1}{s^2+9}\cdot\frac{1}{s^2+9}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+9}\right) = \frac{1}{3}\sin 3t$$

$$\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{1}{s^2+9}\cdot\frac{1}{s^2+9}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s^2+9}\right) * \mathcal{L}^{-1}\left(\frac{1}{s^2+9}\right)$$

9 continue...

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+9}\right)\cdot\mathcal{L}^{-1}\left(\frac{1}{s^2+9}\right)$$

$$= \int_0^t \frac{1}{9}\sin 3\tau\cdot\sin 3(t-\tau)\,d\tau$$

$$= \frac{1}{18}\int_0^t \cos(3t-6\tau) - \cos 3t\,d\tau$$

$$= \frac{1}{18}\left[-\frac{1}{6}\sin(3t-6\tau) - \tau\cos 3t\right]\Bigg|_{\tau=0}^{\tau=t}$$

$$= \frac{1}{18}\left[-\frac{1}{6}\sin(-3t) - t\cos 3t + \frac{1}{6}\sin(3t)\right]$$

$$= \frac{1}{18}\left[\frac{1}{3}\sin 3t - t\cos 3t\right]$$

15. $f(t) = t\sin 3t$ (use Theorem 2, P_{299})

$$g(t) = \sin 3t$$

$$G(s) = \mathcal{L}(g(t)) = \frac{3}{s^2+9}$$

$$f(t) = t\cdot g(t)$$

$$\mathcal{L}(f(t)) = \mathcal{L}(t\cdot g(t)) = -G'(s)$$

$$= \frac{6s}{(s^2+9)^2}$$

Thus $F(s) = \dfrac{6s}{(s^2+9)^2}$

19. $f(t) = \dfrac{\sin t}{t}$

$$\mathcal{L}(\sin t) = \frac{1}{s^2+1}$$

$$\mathcal{L}\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{\sigma^2+1}\,d\sigma = \arctan\sigma\Big|_s^\infty$$

$$= \frac{\pi}{2} - \arctan s$$

28. $F(s) = \dfrac{s}{(s^2+1)^3}$

Theorem 3 P301

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^3}\right\} = t\,\mathcal{L}^{-1}\left\{ \int_s^\infty \frac{\sigma}{(\sigma^2+1)^3}\,d\sigma \right\}$$

$$= t\,\mathcal{L}^{-1}\left\{ -\frac{1}{4}\frac{1}{(\sigma^2+1)^2}\Bigg|_{\sigma=s}^{\sigma=\infty} \right\}$$

$$= t\,\mathcal{L}^{-1}\left\{ \frac{1}{4}\frac{1}{(s^2+1)^2} \right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\cdot\frac{1}{s^2+1}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) * \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= \sin t * \sin t$$

$$= \int_0^t \sin\tau \cdot \sin(t-\tau)\,d\tau$$

$$= \frac{1}{2}\int_0^t \cos(t-2\tau) - \cos t \; d\tau$$

$$= \frac{1}{2}\left[-\frac{1}{2}\sin(t-2\tau) - \tau\cos t\right]\Bigg|_0^t$$

$$= \frac{1}{2}(\sin t - t\cos t)$$

Thus

$$\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^3}\right) = t\cdot\frac{1}{4}\left[\frac{1}{2}(\sin t - t\cos t)\right]$$

$$= \frac{t}{8}\sin t - \frac{1}{8}t^2\cos t$$

29.

$$t x'' + (t-2)x' + x = 0, \quad x(0) = 0$$

(Note that the question does not give the value of $x'(0)$, we need to find it)

when $t=0$, the equation becomes

$$0\cdot x''(0) + (0-2)x'(0) + x(0) = 0$$

$$\Rightarrow x'(0) = 0$$

$$\mathcal{L}(x') = s\,\mathcal{L}(x) - x(0) = sX(s)$$

$$\mathcal{L}(tx') = (-1)\frac{d}{ds}(sX(s))$$

$$= (-1)\left[X(s) + sX'(s)\right]$$

$$\mathcal{L}(x'') = s^2\mathcal{L}(x) - s\,x(0) - x'(0)$$

$$= s^2X(s)$$

$$\mathcal{L}(tx'') = (-1)\frac{d}{ds}(s^2X(s))$$

$$= (-1)\left[2sX(s) + s^2X'(s)\right]$$

plug back into equation

$$(-1)\left[2sX(s) + s^2X'(s)\right] + (-1)\left[X(s) + sX'(s)\right] - 2sX(s) + X(s) = 0$$

$$\Rightarrow \frac{X'(s)}{X(s)} = \frac{4}{-(s+1)}$$

solve by seperation of variable

$$\Rightarrow X(s) = \frac{C}{(s+1)^4}$$

$$x(t) = \mathcal{L}^{-1}(X(s)) = \mathcal{L}^{-1}\left(\frac{C}{(s+1)^4}\right) = Ct^3e^{-t}$$

§ 4.5

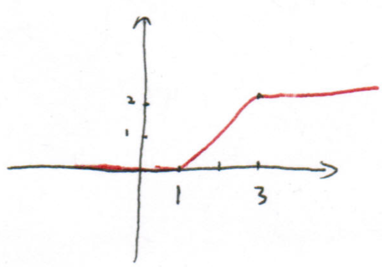
2. $F(s) = \dfrac{e^{-s} - e^{-3s}}{s^2}$

$\mathcal{L}^{-1}\left(\dfrac{e^{-s}}{s^2}\right) = u(t-1)\cdot(t-1)$

$\mathcal{L}^{-1}\left(\dfrac{e^{-3s}}{s^2}\right) = u(t-3)(t-3)$

$f(t) = u(t-1)\cdot(t-1) - u(t-3)\cdot(t-3)$

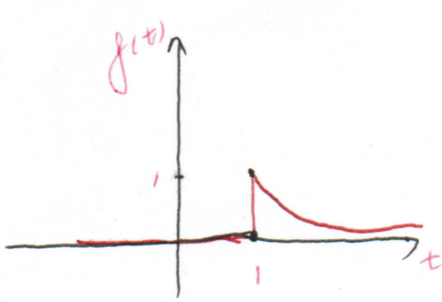
$$= \begin{cases} 0 & t<1 \\ t-1 & 1 \le t \le 3 \\ 2 & t \geq 3 \end{cases}$$



3. $F(s) = \dfrac{e^{-s}}{s+2}$, $\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = e^{-2t}$

$\mathcal{L}^{-1}\left(\dfrac{e^{-s}}{s+2}\right) = u(t-1)\cdot e^{-2(t-1)}$

$$f(t) = u(t-1)\,e^{-2(t-1)} = \begin{cases} 0 & t<1 \\ e^{-2(t-1)} & t \geq 1 \end{cases}$$



11. $f(t)=2$ if $0 \le t < 3$; $f(t) = 0$ if $t \geq 3$

$$f(t) = \begin{cases} 2 & 0 \le t < 3 \\ 0 & t \geq 3 \end{cases}$$

$= 2\left[1 - u(t-3)\right]$

$\mathcal{L}(f(t)) = \mathcal{L}(2 - 2u(t-3))$

$= \dfrac{2}{s} - \dfrac{2e^{-3s}}{s}$

25. $f(t) = 1 - u(t-a)$ $\quad 0<t<2a$

with period $p=2a$, by theorem 2, P_{318}

$F(s) = \dfrac{1}{1-e^{-2as}} \displaystyle\int_0^{2a} e^{-st} f(t)\, dt$

$= \dfrac{1}{1-e^{-2as}} \displaystyle\int_0^{a} e^{-st}\, dt$

$= \dfrac{1}{1-e^{-2as}} \cdot \dfrac{1-e^{-as}}{s}$

$= \dfrac{1}{1+e^{-as}} \cdot \dfrac{1}{s}$

§4.6

1. $x'' + 4x = \delta(t)$; $x(0) = x'(0) = 0$

$$\mathcal{L}(x'') = s^2\mathcal{L}(x) - \cancel{s2}\, s\cdot x'(0) - x(0)$$

$$= s^2 X(s)$$

$\mathcal{L}(\delta(t)) = 1$

The transform equation is

$$s^2 X(s) + 4X(s) = 1$$

$$X(s) = \frac{1}{s^2+4}$$

$$x(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2}\sin(2t)$$

9. from 1 we know $w(t) = \frac{1}{2}\sin(2t)$

$$x(t) = \int_0^t \frac{1}{2}\sin 2\tau \cdot f(t-\tau)\, d\tau.$$