Part I

a) $x'' + 3x' + 7x = t^2$

define $x_1 = x$, $x_2 = x' = x_1'$ then $x_2' + 3x_2 + 7x_1 = t^2 \Rightarrow x_2' = -3x_2 - 7x_1 + t^2$

Hence we have

$$\begin{cases} x_1' = x_2 \\ x_2' = -7x_1 - 3x_2 + t^2 \end{cases} \Leftrightarrow \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -7 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ +t^2 \end{bmatrix}$$

b) $x^{(4)} + 6x'' - 3x' + x = \cos 3t$

define $x_1 = x$, $x_2 = x' = x_1'$, $x_3 = x'' = x_2'$, $x_4 = x''' = x_3'$, substitute into equation, then $x_4' + 6x_3 - 3x_2 + x_1 = \cos 3t \Rightarrow x_4' = -6x_3 + 3x_2 - x_1 + \cos 3t$

Hence

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_4 \\ x_4' = -6x_3 + 3x_2 - x_1 + \cos 3t \end{cases} \Leftrightarrow \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 3 & -6 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cos 3t \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Part II.

5.2

3. $\begin{cases} x' = -3x + 2y & \cdots\ ① \\ y' = -3x + 4y & \cdots\ ② \end{cases}$

$① \Rightarrow y = \dfrac{x' + 3x}{2}$ substitute into ② we have $\dfrac{x'' + 3x'}{2} = -3x + 4\left(\dfrac{x' + 3x}{2}\right)$

$\Rightarrow \quad x'' - x' - 6x = 0$,

characteristic equation $\quad r^2 - r - 6 = 0 \Rightarrow r = 3,\ r = -2$

so its general solution is

$$x(t) = C_1 e^{3t} + C_2 e^{-2t}$$,

then $y = \frac{x' + 3x}{2} = 3C_1 e^{3t} + \frac{1}{2} C_2 e^{-2t}$

The initial condition is $\begin{cases} x(0) = 0 \\ y(0) = 2 \end{cases} \Rightarrow \begin{cases} C_1 + C_2 = 0 \\ 3C_1 + \frac{1}{2} C_2 = 2 \end{cases} \Rightarrow \begin{cases} C_1 = \frac{4}{5} \\ C_2 = -\frac{4}{5} \end{cases}$

Hence $\quad \begin{cases} x = \frac{4}{5} e^{3t} - \frac{4}{5} e^{-2t} \\ y = \frac{12}{5} e^{3t} - \frac{2}{5} e^{-2t} \end{cases}$

23、

$$(D+2)x + (D+2)y = e^{-3t}$$

$$(D+3)x + (D+3)y = e^{-2t}$$

$L = \begin{pmatrix} D+2 & D+2 \\ D+3 & D+3 \end{pmatrix} \qquad |L| = 0 \Rightarrow$ the system is degenerate

$$\begin{vmatrix} e^{-3t} & D+2 \\ e^{-2t} & D+3 \end{vmatrix} = (D+3)e^{-3t} - (D+2)e^{-2t}$$

$$= -3e^{-3t} + 3e^{-3t} - (-2e^{-2t} + 2e^{-2t})$$

$$= 0$$

Hence the system has infinite many solution

5.3

13. $\begin{cases} x' = 2x + 4y + 3e^t \\ y' = 5x - y - t^2 \end{cases}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3e^t \\ -t^2 \end{bmatrix}$$

20. $\begin{cases} x_1' = x_2 + x_3 + 1 \\ x_2' = x_3 + x_4 + t \\ x_3' = x_1 + x_4 + t^2 \\ x_4' = x_1 + x_2 + t^3 \end{cases} \Rightarrow \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$

5.4

6. $\begin{cases} x_1' = 9x_1 + 5x_2 \\ x_2' = -6x_1 - 2x_2 \end{cases} \Rightarrow \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

define $A = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix}$

$$\det(A - \lambda I) = (9-\lambda)(-2-\lambda) - (-6)\cdot 5 = \lambda^2 - 7\lambda + 12 = (\lambda-3)(\lambda-4) = 0$$

$\lambda = 3$ or $\lambda = 4$

then we need to find eigenvector

~~then~~ $(A - 3I)V = 0 \Leftrightarrow \begin{bmatrix} 6 & 5 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \Rightarrow v = b \begin{bmatrix} -\frac{5}{6} \\ 1 \end{bmatrix}$

choose $V_1 = \begin{bmatrix} -\frac{5}{6} \\ 1 \end{bmatrix}$

$$(A-4I)V=0 \iff \begin{pmatrix} 5 & 5 \\ -6 & -6 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 0 \implies v = b\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Choose $v_2 := \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Hence the general solution is

$$\vec{x} = c_1\vec{v_1}e^{3t} + c_2 v_2 e^{4t} = \begin{bmatrix} -\frac{5}{6}c_1e^{3t} - c_2e^{4t} \\ c_1e^{3t} + c_2e^{4t} \end{bmatrix}$$

$$\begin{matrix} x_1(0) = 1 \\ x_2(0) = 0 \end{matrix} \implies \begin{matrix} -\frac{5}{6}c_1 - c_2 = 1 \\ c_1 + c_2 = 0 \end{matrix} \implies \begin{cases} c_1 = 6 \\ c_2 = -6 \end{cases}$$

Thus the general solution is

$$\vec{x} = \begin{bmatrix} -5e^{3t} + 6e^{4t} \\ 6e^{3t} - 6e^{4t} \end{bmatrix}$$

26.

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 3 & 0 & 1 \\ 9 & -1 & 2 \\ -9 & 4 & -1 \end{pmatrix}\vec{x}$$

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 9 & -1-\lambda & 2 \\ -9 & 4 & -1-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} -1-\lambda & 2 \\ 4 & -1-\lambda \end{vmatrix} - 0\begin{vmatrix} 9 & 2 \\ -9 & -1-\lambda \end{vmatrix} + 1\cdot\begin{vmatrix} 9 & -1-\lambda \\ -9 & 4 \end{vmatrix}$$

$$= (3-\lambda)\left[(1+\lambda)^2 - 8\right] + \left(36 - 9(1+\lambda)\right) = \cancel{(3-\lambda)(\lambda-3)(\lambda+5) - 9(\lambda+5)}$$

$$= (3-\lambda)\left[(1+\lambda)^2 + 1\right]$$

$$\det(A-\lambda I)=0 \implies \lambda=3 \quad \text{or} \quad \lambda=-1\pm i$$

i) $(A-3I)v=0 \implies \begin{pmatrix} 0 & 0 & 1 \\ 9 & -4 & 2 \\ -9 & 4 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=0 \implies \begin{cases} a=\frac{4}{9}b \\ c=0 \end{cases} \implies v=b\begin{bmatrix} \frac{4}{9} \\ 1 \\ 0 \end{bmatrix}$

Choose $v_1=\begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix}$.

ii) $(A-(-1+i)I)v=0 \implies \begin{pmatrix} 4-i & 0 & 1 \\ 9 & -i & 2 \\ -9 & 4 & -i \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=0 \implies \begin{cases} b=(2-i)a \\ c=(i-4)a \end{cases} \implies v=a\begin{bmatrix} 1 \\ 2-i \\ i-4 \end{bmatrix}$

Choose $\vec{v}=\begin{bmatrix} 1 \\ 2-i \\ i-4 \end{bmatrix}$ then $\vec{v}\cdot e^{(-1+i)t}=\begin{bmatrix} 1 \\ 2-i \\ i-4 \end{bmatrix}e^{-t}(\cos t+i\sin t)$

$$=\begin{bmatrix} \cos t+i\sin t \\ 2\cos t+\sin t+i(-\cos t+2\sin t) \\ -4\cos t-\sin t+i(\cos t-4\sin t) \end{bmatrix}e^{-t}.$$

Hence $\vec{X_2}=\cancel{\text{Real}}\ \mathrm{Re}(\vec{v}e^{(-1+i)t})=\begin{bmatrix} \cos t \\ 2\cos t+\sin t \\ -4\cos t-\sin t \end{bmatrix}e^{-t}$

$$\vec{X_3}=\mathrm{Im}(\vec{v}e^{(-1+i)t})=\begin{bmatrix} \sin t \\ -\cos t+2\sin t \\ \cos t-4\sin t \end{bmatrix}e^{-t}$$

Thus the general solution is

$$\vec{x} = c_1\vec{v}_1 e^{3t} + c_2\vec{x}_2 + c_3\vec{x}_3$$

$$= \begin{bmatrix} 4c_1 \\ 9c_1 \\ 0 \end{bmatrix} e^{3t} + \begin{bmatrix} c_2 e^{-t}\cos t \\ 2c_2 e^{-t}\left(\cos t + \frac{\sin t}{2}\right) \\ c_2 e^{-t}(-4\cos t - \sin t) \end{bmatrix} + \begin{bmatrix} c_3 \sin t \\ c_3(-\cos t + 2\sin t) \\ c_3(\cos t - 4\sin t) \end{bmatrix} e^{-t}$$

The initial condition $\vec{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 17 \end{bmatrix}$ yields.

$$\begin{cases} 4c_1 + c_2 = 0 \\ 9c_1 + 2c_2 - c_3 = 0 \\ -4c_2 + c_3 = 17 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = -4 \\ c_3 = 1 \end{cases}$$

Hence the particular solution of the system is

$$\begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix} e^{3t} - 4\begin{bmatrix} \cos t \\ 2\left(\cos t + \frac{\sin t}{2}\right) \\ -4\cos t - \sin t \end{bmatrix} e^{-t} + \begin{bmatrix} \sin t \\ -\cos t + 2\sin t \\ \cos t - 4\sin t \end{bmatrix} e^{-t}$$

5.6

$$\vec{x}' = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \vec{x}$$

$$A = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \qquad \det(A - \lambda I) = \begin{vmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{vmatrix} = (7-\lambda)(3-\lambda) + 4 = (\lambda-5)^2$$

$\lambda = 5$ with multiplicity 2.

$$(A - 5I)\, v = 0 \Rightarrow \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow b = -2a \Rightarrow v = a \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

only 1 independent vector.

Hence the general solution is

$$\vec{x} = c_1 \vec{V_1} e^{5t} + c_2 (\vec{V_1} t + \vec{V_2}) e^{5t}$$

$\vec{V_2}$ satisfies $(A - 5I)^2 V_2 = 0 \Leftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow V_2$ Can be any vector.

choose $\vec{V_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

then $\vec{V_1} = (A - 5I) V_2 = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

Thus

$$\vec{x} = c_1 \begin{bmatrix} 2 \\ -4 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 1+2t \\ -4t \end{bmatrix} e^{5t}$$

12、 $\vec{x}' = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \vec{x}$

$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$. $\quad \det(A-\lambda I) = \begin{vmatrix} -1-\lambda & 0 & 1 \\ 0 & -1-\lambda & 1 \\ 1 & -1 & -1-\lambda \end{vmatrix} = -(1+\lambda)^3$

$\lambda = -1$ with multiplicity 3 .

$(A+I)\vec{v} = 0 \iff \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow \begin{cases} a=b \\ c=0 \end{cases} \Rightarrow v = b\cdot\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

only 1 independent variable.

Hence the general solution is $\vec{x} = c_1\vec{v}_1 e^{-t} + (\vec{v}_1 t + \vec{v}_2) c_2 e^{-t} + (\frac{\vec{v}_1 t^2}{2} + \vec{v}_2 t + \vec{v}_3) c_3 e$

$(A+I)^3 \vec{V}_3 = 0 \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow \vec{V}_3$ Can be any vector independe

of $[1,1,0]^T$.

Choose $\vec{V}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

then $\vec{V}_2 = (A+I)V_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\vec{V}_1 = (A+I)V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Thus the general solution is

$$\vec{x} = C_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} t \\ t \\ 1 \end{bmatrix} e^{-t} + C_3 \begin{bmatrix} \frac{t^2}{2}+1 \\ \frac{t^2}{2} \\ t \end{bmatrix} e^{-t}$$

5.5.7.

1. $$\vec{x}\,' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$\det(A-\lambda I) = (2-\lambda)(2-\lambda)-1 = (\lambda-1)(\lambda-3) = 0 \Rightarrow \lambda=1 \text{ or } \lambda=3$$

$$(A-I)\,v = 0 \Leftrightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 \Rightarrow a=-b \Rightarrow v = b\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Choose $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$(A-3I)\,v = 0 \Leftrightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 \Rightarrow a=b \Rightarrow v = b\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Choose $V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Hence $\vec{x}_1 = \vec{v}_1 e^{t} = \begin{bmatrix} -e^{t} \\ e^{t} \end{bmatrix}$ $\quad \vec{x}_2 = \vec{v}_2 e^{3t} = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$

$$\Phi = [\vec{x}_1, \vec{x}_2] = \begin{bmatrix} -e^{t} & e^{3t} \\ e^{t} & e^{3t} \end{bmatrix}, \quad \Phi(0) = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \Phi(0)^{-1} = -\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

$$\vec{x}(t) = \Phi(t)\,\Phi(0)^{-1}\cdot x_0 = \begin{bmatrix} -e^{t} & e^{3t} \\ e^{t} & e^{3t} \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2}e^{t} + \frac{1}{2}e^{3t} \\ -\frac{5}{2}e^{t} + \frac{1}{2}e^{3t} \end{bmatrix}$$

9.

$$\begin{cases} x_1' = 5x_1 - 4x_2 \\ x_2' = 2x_1 - x_2. \end{cases} \iff \frac{d\vec{x}}{dt} = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix} \vec{x}$$

$$\det(A-\lambda I) = (5-\lambda)(-1-\lambda)+8 = \overset{(\lambda-3)(\lambda-1)}{\cancel{(\lambda+3)(\lambda+1)}} = 0 \Rightarrow \lambda = +3 \text{ or } \lambda = +1.$$

$$(A-3I)\,v = \begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \Rightarrow a = 2b \Rightarrow v = b\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

choose $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$(A-I)\,v = \begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \Rightarrow a = b \Rightarrow v = b\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

choose $V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$x_1(t) = \vec{V_1} e^{3t} = \begin{bmatrix} 2e^{3t} \\ e^{3t} \end{bmatrix} \qquad x_2 = \vec{V_2} e^{t} = \begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix}$$

Then $\Phi(t) = \begin{bmatrix} 2e^{3t} & e^{t} \\ e^{3t} & e^{t} \end{bmatrix}$ $\quad \Phi(0) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad \Phi(0)^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

$$e^{At} = \Phi(t)\,\Phi(0)^{-1} = \begin{bmatrix} 2e^{3t}-e^{t}, & -2e^{3t}+2e^{t} \\ e^{3t}-e^{t}, & -e^{3t}+2e^{t} \end{bmatrix}$$

21. $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = 0 \Rightarrow A \text{ is nilponent}$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^n t^n}{n!} + \cdots$$

$$= I + At + 0$$

$$= \begin{bmatrix} 1+t & -t \\ t & \cancel{1+t} \; 1-t \end{bmatrix}$$

5.5.8

19.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)(2-\lambda) - 4 = \lambda(\lambda-4) = 0 \Rightarrow \lambda = 0 \text{ or } \lambda = 4$$

$$AV_1 = 0 \Rightarrow \vec{V_1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_c(t) \\ y_c(t) \end{bmatrix} = c_1\vec{V_1} + c_2\vec{V_2}e^{4t}$$

$$(A - 4I)V_2 = 0 \Rightarrow \vec{V_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Hence we select

$$\begin{bmatrix} x_p(t) \\ y_p(t) \end{bmatrix} = \vec{a}t + \vec{b}$$

Upon substitution of $\begin{bmatrix} x_p \\ y_p \end{bmatrix}$ into equation system. we get

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2a_1 + 4a_2 \\ a_1 + 2a_2 \end{bmatrix} t + \begin{bmatrix} 2b_1 + 4b_2 + 2 \\ b_1 + 2b_2 + 3 \end{bmatrix}$$

equating the coefficent of t.

$$\begin{cases} 2a_1 + 4a_2 = 0 \\ a_1 + 2a_2 = 0 \\ a_1 = 2b_1 + 4b_2 + 2 \\ a_2 = b_1 + 2b_2 + 3 \end{cases} \Rightarrow \begin{cases} a_1 = -2 \\ a_2 = 1 \\ b_1 + 2b_2 = -2 \end{cases}$$ we can choose $\begin{matrix} b_1 = -2 \\ b_2 = 0 \end{matrix}$

Then the general solution is

$$\begin{bmatrix} \vec{x}_p \\ y \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} + \begin{bmatrix} -2t - 2 \\ t \end{bmatrix}$$

$$\left.\begin{matrix} x(0) = 1 \\ y(0) = -1 \end{matrix}\right\} \Rightarrow \begin{cases} -2c_1 - 2 + 2c_2 = 1 \\ c_1 + c_2 = -1 \end{cases} \Rightarrow \begin{cases} c_1 = -\frac{5}{4} \\ c_2 = \frac{1}{4} \end{cases}$$

Hence

$$x = \frac{1}{2} + \frac{1}{2} e^{4t} - 2t$$

$$y = -\frac{5}{4} + \frac{1}{4} e^{4t} + t$$

23. We need to use the formula $x(t) = e^{A(t-a)} x_a + e^{At} \int_a^t e^{-As} f(s)\, ds$

$$e^{-At} f(t) = \begin{bmatrix} 1-3t & t \\ -9t & 1+3t \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 7-16t \\ 5-48t \end{bmatrix}$$

$$\int_0^t e^{-As} f(s)\, ds = \begin{bmatrix} 7t-8t^2 \\ 5t-24t^2 \end{bmatrix}$$

$$e^{At} \int_0^t e^{-As} f(s)\, ds = \begin{bmatrix} 1+3t & -t \\ 9t & 1-3t \end{bmatrix} \begin{bmatrix} 7t-8t^2 \\ 5t-24t^2 \end{bmatrix} = \begin{bmatrix} 7t+8t^2 \\ 5t+2t^2 \end{bmatrix}$$

$$e^{At} \cdot x(0) = \begin{bmatrix} 1+3t & -t \\ 9t & 1-3t \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3+4t \\ 5+12t \end{bmatrix}$$

Hence $x(t) = e^{At} + e^{At} \int_0^t e^{-As} f(s)\, ds = \begin{bmatrix} 8t^2+11t+3 \\ 24t^2+17t+5 \end{bmatrix}$