1. (a) $\frac{dy}{dx} = y\sin x$

$\int \frac{dy}{y} = \int \sin x \, dx$

$\ln|y| = -\cos x + C_1$

$y = C \cdot e^{-\cos x}$

+8

(b) $2xy' - 3y = 9x^3$

$y' - \frac{3}{2x} y = \frac{9}{2} x^2$

$v = e^{-\frac{3}{2}\int \frac{1}{x} dx} = e^{-\frac{3}{2}\ln|x|} = x^{-\frac{3}{2}}$

$(x^{-\frac{3}{2}} \cdot y)' = \frac{9}{2} x^{\frac{1}{2}}$

$x^{-\frac{3}{2}} \cdot y = \frac{9}{2}\int x^{\frac{1}{2}} dx = \frac{9}{2} \cdot \frac{2}{3} \cdot x^{\frac{3}{2}} + C$ +8

$y = 3x^3 + Cx^{\frac{3}{2}}$

(c) $xyy' = x^2 + 2y^2$

$y' = \frac{x}{y} + \frac{2y}{x}$

let $v = \frac{y}{x}$, $y = v \cdot x$, $y' = v' \cdot x + v$

so $v'x + v = \frac{1}{v} + 2v$

$\frac{dv}{dx} \cdot x = \frac{1+v^2}{v}$

$\int \frac{v}{1+v^2} dv = \int \frac{dx}{x}$

$\frac{1}{2}\ln|1+v^2| = \ln|x| + C_1$

$\ln|1+v^2| = \ln x^2 + 2C_1$ +8

$1 + v^2 = C \cdot x^2$

$\frac{y^2}{x^2} = Cx^2 - 1$

$y^2 = Cx^4 - x^2$

(d) $xy' + 6y + 3xy^{\frac{4}{3}} = 0$

$$y' + \frac{6}{x}y = -3y^{\frac{4}{3}} \quad (n = \tfrac{4}{3})$$

let $v = \frac{1}{y^{\frac{1}{3}}} = y^{-\frac{1}{3}}$, $v' = -\frac{1}{3}y^{-\frac{4}{3}} \cdot y'$

$$(-\tfrac{1}{3})\, y^{-\frac{4}{3}} \cdot y' + \frac{(-2)}{x} y^{-\frac{1}{3}} = 1$$

$$v' - \frac{2}{x} \cdot v = 1$$

integration factor $= e^{-\int \frac{2}{x}dx} = e^{-2\ln|x|} = x^{-2}$

$$(x^{-2} \cdot v)' = x^{-2}$$

$$x^{-2} \cdot v = \int x^{-2}dx = -x^{-1} + C$$

$$v = -x + Cx^2$$

So $y^{-\frac{1}{3}} = -x + Cx^2$

$$y = (-x + Cx^2)^{-3} = \frac{1}{(-x + Cx^2)^3}$$

+8

(e) $x^2y'' + 3xy' = 2$

let $v = y'$, $y'' = v'$

$$x^2 \cdot v' + 3x \cdot v = 2$$

$$v' + \frac{3}{x}v = \frac{2}{x^2}$$

integration factor $= e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$

$$(x^3 \cdot v)' = 2x$$

$$x^3 \cdot v = 2\int x\,dx = x^2 + C$$

$$v = \frac{1}{x} + C \cdot x^{-3}$$

So $\frac{dy}{dx} = \frac{1}{x} + Cx^{-3}$

$$y = \ln|x| + C_1 x^{-2} + C_2$$

+8

2. $3y^{(3)} + 2y'' = 0$

$3r^3 + 2r^2 = 0$

$r_1 = r_2 = 0, \quad r_3 = -\frac{2}{3}$

general solution $y = C_1 + C_2 t + C_3 e^{-\frac{2}{3}t}$

$y(0) = -1 \qquad C_1 + C_3 = -1$

$y'(0) = 0 \qquad C_2 - \frac{2}{3}C_3 = 0$

$y''(0) = 1 \qquad \frac{4}{9}C_3 = 1 \Rightarrow C_3 = \frac{9}{4}$

$\Rightarrow C_2 = \frac{3}{2}, \quad C_1 = -\frac{13}{4}$

+10

So the solution is

$$y = -\frac{13}{4} + \frac{3}{2}t + \frac{9}{4}e^{-\frac{2}{3}t}$$

3. $y'' + 9y = \sin(3x)$

$r^2 + 9 = 0 \Rightarrow r = \pm 3i$

$y_c = C_1 \cos 3x + C_2 \sin 3x$

$y_p = x \cdot (A \sin(3x) + B\cos(3x))$

$y_p' = A\sin 3x + 3Ax\cos 3x + B\cos(3x) - 3Bx\sin 3x$

$= (A - 3Bx)\sin 3x + (B + 3Ax)\cos 3x$

$y_p'' = -3B\sin 3x + 3(A - 3Bx)\cos 3x + 3A\cos 3x - 3(B + 3Ax)\sin 3x$

$= (-9Ax - 6B)\sin 3x + (6A - 9Bx)\cos 3x$

so $(-9Ax - 6B)\sin 3x + (6A - 9Bx)\cos 3x + 9Ax\sin 3x + 9Bx\cos 3x = \sin 3x$

$$\begin{cases} A = 0 \\ -6B = 1 \end{cases} \Rightarrow B = -\frac{1}{6}$$

✓ +10

so $y_p = -\frac{1}{6}x\cos 3x$

and thus the solution is

$$y = C_1\cos 3x + C_2 \sin 3x - \frac{1}{6}x\cos 3x$$

4. $y'' + 9y = 2\sec 3x$

$r^2 + 9 = 0 \Rightarrow r = \pm 3i$

$y_c = C_1 \cos 3x + C_2 \sin 3x$, so $y_1 = \cos 3x$, $y_2 = \sin 3x$

$$W(x) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3$$

$$y_p = -\cos 3x \int \frac{2\sin 3x}{3\cos 3x}\,dx + \sin 3x \int \frac{2}{3}\,dx$$

$$= -\frac{2}{3}\cos 3x \int \frac{\sin 3x}{\cos 3x}\,dx + \frac{2}{3} x \sin 3x$$ +10

$$= \frac{2}{9}\cos 3x \cdot \ln|\cos 3x| + \frac{2}{3} x \sin 3x$$

5. $y'' + \lambda y = 0$

$r^2 + \lambda = 0$

if $\lambda = 0$, $r^2 = 0$ $\quad y = Ax + B$, $y' = A$

to have $y'(-\pi) = y'(\pi) = 0$ $\quad A = 0$

so $\lambda = 0$ is an eigenvalue, in this case $y = B$ (a constant) +15

if $\lambda \neq 0$, let $\lambda = \alpha^2$

- $\lambda > 0$ $\quad r^2 = -\lambda \Rightarrow r = \pm \alpha i \Rightarrow y = A\cos\alpha t + B\sin\alpha t$

$y' = -\alpha A \sin\alpha t + \alpha B \cos\alpha t$

$$\begin{cases} \alpha A \sin\alpha\pi + \alpha B \cos\alpha\pi = 0 \\ -\alpha A \sin\alpha\pi + \alpha B \cos\alpha\pi = 0 \end{cases} \Rightarrow \begin{cases} A\sin\alpha\pi + B\cos\alpha\pi = 0 \\ -A\sin\alpha\pi + B\cos\alpha\pi = 0 \end{cases}$$

if $A = 0$, $B \neq 0$, then $\cos\alpha\pi = 0$ $\quad \alpha = \frac{\pi}{2} + (k-1)$

in this case $y = B\sin\frac{k}{2}t$

eigenfunction is $\sin\frac{k}{2}t$

if $A \neq 0$, $B = 0$, $\sin\alpha\pi = 0$. $\quad \alpha = k$

in this case $y = A\cos kt$

eigenfunction is $\cos kt$

6. $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\} = \mathcal{L}^{-1}\left\{\left(\frac{s}{s^2+4}\right)\cdot\left(\frac{s}{s^2+4}\right)\right\}$

we know $\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t$

so $\mathcal{L}^{-1}\left\{\left(\frac{s}{s^2+4}\right)\left(\frac{s}{s^2+4}\right)\right\} = \int_0^t \cos 2(t-u)\cos 2u\,du$

$= \int_0^t \frac{1}{2}(\cos 2t + \cos(2t-4u))\,du$

$= \frac{1}{2}t\cos 2t + \frac{1}{2}\int_0^t \cos(2t-4u)\,du$

$= \frac{1}{2}t\cos 2t - \frac{1}{2}\cdot\frac{1}{4}\sin(2t-4u)\Big|_0^t$

$= \frac{1}{2}t\cos 2t + \frac{1}{4}\sin 2t$

so $f(t) = \frac{1}{2}t\cos 2t + \frac{1}{4}\sin 2t$

+10

7. $x'' + x = \sin 2t$

$s^2F(s) + F(s) = \frac{2}{s^2+4}$

$F(s) = \frac{2}{(s^2+4)(s^2+1)} = \frac{\left(-\frac{2}{3}\right)}{s^2+4} + \frac{\left(\frac{2}{3}\right)}{s^2+1} = \left(-\frac{1}{3}\right)\frac{2}{s^2+4} + \left(\frac{2}{3}\right)\frac{1}{s^2+1}$

$x(t) = \mathcal{L}^{-1}\{F(s)\} = -\frac{1}{3}\sin 2t + \frac{2}{3}\sin t$

$= -\frac{1}{3}(\sin 2t - 2\sin t)$

+10

8. let $t=0 \Rightarrow -X'(0)+3X(0)=0 \Rightarrow X'(0)=0$

$\mathcal{L}\{X''\} = S^2 F(s)$

$\mathcal{L}\{X'\} = S F(S)$

$tX'' + 3tX' - X' + 3X = 0$

take laplace transform on both sides

$-\frac{d}{ds}(S^2 F(S)) - 3\cdot\frac{d}{ds}(S\cdot F(S)) - SF(S) + 3F(S) = 0$

$-(2S\cdot F(S) + S^2F'(S)) - 3(F(S)+SF'(S)) - SF(S) + 3F(S) = 0$

$(-S^2-3S)F'(S) + (-2S-3-S+3)F(S) = 0$

$(S^2+3S)F'(S) = (-2S-S)F(S)$

$(S^2+3S)\frac{dF}{ds} = (-3S)F$

$\int \frac{dF}{F} = \int \frac{-3}{S+3}\,ds$

$\ln|F(S)| = -3\ln|S+3| + C_1$

$F(S) = C\cdot(S+3)^{-3}$

+15

$X(t) = \mathcal{L}^{-1}\{F(S)\} = \mathcal{L}^{-1}\{\frac{1}{(S+3)^3}\}$

$= \frac{1}{2}e^{-3t}t^2$