second order linear equations with constant coeffient

$$y'' + Ay' + By = C$$

General solution $y = c_1 y_1 + c_2 y_2$ initial condition are satisfied by choosing C_1 and C_2 properly.

where y_1 and y_2 are solutions

The task of solving this equation is reduced to just find two solutions.

where do we get these solutions?

Task: To find 2 independent solution.

Try $y = e^{rt}$, Then $y'' = r^2 e^{rt}$. $y' = re^{rt}$

plug in

$$r^2 e^{rt} + Are^{rt} + Be^{rt} = 0$$

$$\Longleftrightarrow \quad r^2 + Ar + B = 0$$ ← Characteristic equation.

Solve for r, there are three cases about the root.

- real and unequal $(r_1 \neq r_2)$

- equal

- Complex root.

Case 1: Example 1: $y''+4y'+3y=0$.

The characteristic equation is: $r^2+4r+3=0$

solve it : $r_1=-1$, $r_2=-3$

Thus the general solution is: $y=c_1e^{-t}+c_2e^{-3t}$

Case 2: Example 2: $y''+2y'+y=0$

The characteristic equation is $r^2+2r+1=0$

Solve it : $r=-1$.

Thus the general solution is: $y=(c_1+c_2t)e^{-t}$

Case 3: Example 3: $y''+2y'+2=0$

$r^2+2r+2=0$

$r=-1\pm i$

$y=(c_1\cos t+c_2\sin t)e^{-t}$

③

Theory of General second order homogenous ODEs.

i) why are $C_1y_1 + C_2y_2$ solutions ?

ii) why are they all the solutions ?

Q1: Superprosition principle: (See Theorem 1 in 2.1)

If y_1, y_2 are solutions to linear homogenous ODE

then $C_1y_1 + C_2y_2$ is a solution

Q2: $\{C_1y_1 + C_2y_2\}$ is enough to satify any initial Condition.

proof: If $y(x_0)=a$, $y'(x_0)=b$, $y = C_1y_1 + C_2y_2$

then

$$C_1y_1(x_0) + C_2y_2(x_0) = a$$

$$C_1y_1'(x_0) + C_2y_2'(x_0) = b$$

Solvable for C_1 and C_2 if $\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \neq 0$

$\Longleftrightarrow$

y_1 and y_2 are independent.

Higher order linear equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$

$$y = e^{rt}$$

characteristic equation: $\cancel{a_n y^n}$ $$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0$$

Example 4. $$y^{(4)} - 3y^{(3)} + 3y'' - y' = 0$$

Characteristic equation: $$r^4 - 3r^3 + 3r^2 - r = 0$$

Solve it by factoring $$r(r-1)^3 = 0$$

$r=1$ is a repeated root of multiplicity 3.

Hence the general solution is

$$y = C_0 + (C_1 + C_2 x + C_3 x^2) e^t$$

Example 5,

$$y^{(4)} + y'' = 0$$

Characteristic equation $\quad r^4 + r^2 = 0$.

Solve it : $\quad r = 0 \;,\; r = \pm i$

0 is a repeated root of multiplicity 2 .

Hence the general solution is

$$y = C_0 + C_1 x + C_2 \cos t + C_3 \sin t .$$

Example 6 ,

$$y^{(4)} + 2y'' \cancel{=0} + y = 0$$

Characteristic equation $\quad r^4 + 2r^2 + 1 = 0$

$$(r^2+1)^2 = 0$$

$$\Leftrightarrow (r+i)^2 (r-i)^2 = 0$$

i is a repeated root of multiplicity 2 .

Hence the general solution is $\quad y = C_1 \cos x + C_2 \sin x + t(C_3 \cos x + C_4 \sin x$