Nonhomogeneous Equations.

$$y'' + p(x)\cdot y' + q(x)\,y = f(x) \quad \cdots \quad ①$$

We focus on the case where $p(x)$ and $q(x)$ are constant

the general solution of ① is of the form

$$y = y_c + y_p$$

where y_c is the solution of homogeneous equation, that is

y_c satisfies:

$$y_c'' + p(x)\,y_c' + q(x)\,y = 0$$

y_p is a particular solution of ①.

Remark:

- $y_c = C_1 y_1 + C_2 y_2$
- y_c is called complementary solution.
- we only need to find one particular solution.
- For constant coefficient, we have learned how to find y_c.

Question: How can we find a particular solution?

Method 1: undetermined ~~met~~ coefficient.

Have a guess about the possible form of y_p and determine the constant coefficient.

Candidate for y_p would be a linear combination of $f(x)$ and its derivatives.

The method is useful when $f(x)$ is of special form. (see P_{156} Figure 2.5.1 for detail).

Rule 1:

If no terms appearing in $f(x)$ or in any of its derivatives satisfies the associated homogenous equation, Then y_p is a linear combination of all linear independent such terms and their derivative.

Remark: we can use wronskian to check whether different functions are linear dependent or not.

Example 1.

$$y'' + 3y' + 4y = 3x+2 \quad \cdots \text{①}$$

The characteristic equation

$r^2 + 3r + 4 = 0$ has roots $r = \dfrac{-3 \pm \sqrt{7}i}{2}$

so $y_c = C_1 e^{-\frac{3}{2}x} \cos(\frac{\sqrt{7}}{2}\cancel{x}) + C_2 e^{-\frac{3}{2}} \sin(\frac{\sqrt{7}}{2}x)$

Since $f(x) = 3x+2$ does not satisfies the homogenous equation

we try $y_p = Ax + B$, then $y_p' = A$, $y_p'' = 0$

substitute into ~~the~~ ①. we have

$$0 + 3A + 4(\cancel{A}x + B) = 3x + 2$$

equating the coeffient of x, we have

$$\begin{cases} 4A = 3 \\ 3A + 4B = 2 \end{cases} \Rightarrow \begin{cases} A = \frac{3}{4} \\ B = -\frac{1}{16} \end{cases}$$

Thus $y_p = \frac{3}{4}x - \frac{1}{16}x$

The general solution is

$$y = C_1 e^{-\frac{3}{2}x} \cos(\frac{\sqrt{7}}{2}x) + C_2 e^{-\frac{3}{2}x} \sin(\frac{\sqrt{7}}{2}x) + \frac{3}{4}x - \frac{1}{16}x.$$

Rule 2.

If $f(x)$ ~~Cont~~ or its derivatives contains terms that satisfy the homogenous equation; we try: ~~y~~

$$y_p = x^s \Big[\text{linear combination } \overbrace{\text{of}}^{\text{the duplicated term in}} f(x) \text{ and its derivatives} \Big] + \underline{\text{the rest}}$$

s is smallest (integer) number such that no term in y_p dulplicates a term in y_c.

Example 2

$$y'' + y = \sin x \qquad \cdots \quad ②$$

The characteristic equation is $r^2+1=0$, $r=\pm i$

~~The general solution~~ is so $y_c = \overset{C_1}{A}\sin x + \overset{C_2}{B}\cos x$

Since $f(x)=\sin x$ contains a term "$\sin x$" which ~~satisfies~~ appears in y_c, we try

$$y_p = x(A\sin x + B\cos x)$$

then $y_p' = A(\sin x + x\cos x) + B(\cos x - x\sin x)$

$$y_p'' = (2A - Bx)\cos x - (Ax+2B)\sin x$$

Plug y_p in ② we have

$$2A\cos x - 2B\sin x = \sin x$$

equating the coeffient of $\sin x$ and $\cos x$ we have

$$\begin{cases} 2A = 0 \\ -2B = 1 \end{cases} \Rightarrow B = -\frac{1}{2}, \quad A = 0$$

$$y_p = -\frac{1}{2} x\cos x .$$

Example 3. Determine the appropriate form for a particular solution of

$$y^{(3)} + y'' = 3e^x + 4x^2$$

The characteristic equation : $r^3 + r^2 = 0$,

$$r = 0 , \quad r = -1$$

$$y_c = C_1 + C_2 x + C_3 e^{-x}$$

since $f(x) = 3e^x + 4x^2$, we could try

$$y_p = \underbrace{Ax^2 + Bx + C}_{(a)} + \underbrace{De}_{(b)}^{-x}$$

term (a) contains term $Bx + C$ which appears in y_c , Hence we multiple x^2 to eliminate the dulplication. Hence we should take

$$y_p = x^2(Ax^2 + Bx + C) + De^{-x}$$

Method 2. Variation of parameters: (no condition on $f(x)$)

$$y_p = -y_1(x)\int \frac{y_2(x) f(x)}{W(x)}\,dx + y_2(x)\int \frac{y_1(x) f(x)}{W(x)}\,dx$$

W is the ~~wrongskian~~ Wronskian of y_1 and y_2.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

y_1, and y_2 are two independent solution of associated homogenous equation.

Example 4. $y'' + y = \sin x$

find the particular solution by ~~u~~using Variation of parameters.

$y_1 = \sin x \qquad y_2 = \cos x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -1.$$

$$\int \frac{y_2 f(x)}{W(x)}\,dx = \int \frac{\cos x \cdot \sin x}{-1}\,dx = -\frac{\sin 2x}{4}$$

$$\int \frac{y_1 f(x)}{W(x)}\,dx = \int \frac{\sin^2 x}{-1}\,dx = -\frac{2x - \sin 2x}{4}$$

$$y_p = -y_1(x)\cdot\int \frac{y_2 f(x)}{W(x)}\,dx + y_2(x)\cdot\int \frac{y_1 f(x)}{W(x)}\,dx = -\sin x\cdot\left(\frac{\sin 2x}{4}\right) + \cos x\cdot\left(\frac{-2x-\sin}{4}\right.$$

$$= -\frac{x\cos x}{2} - \frac{\sin x}{4}$$

Endpoint problems and Eigenvalues.

Model: the shape of a quickly spinning jump rope.

Example 5.

$$y'' + \lambda y = 0 \quad , \quad y(0) = 0 \quad , \quad y(\pi) = 0$$

Find λ such that the endpoint problem admit a non-trivial solution? (or ~~find~~ determine eigenvalues and eigenfunction)

i) If $\lambda < 0$, Then the general solution is

$$y(x) = c_1 e^{\sqrt{-\lambda} x} + c_2 e^{-\sqrt{-\lambda} x}$$

The Boundary condition yields.

$$\left. \begin{array}{l} c_1 + c_2 = 0 \\ c_1 e^{\sqrt{-\lambda}\pi} + c_2 e^{-\sqrt{-\lambda}\pi} = 0 \end{array} \right\} \Rightarrow c_1 = c_2 = 0$$

ii) If $\lambda = 0$ Then the general solution is

$$y(x) = c_1 + c_2 x$$

The boundary condition yields

$$\left. \begin{array}{l} c_1 = 0 \\ c_1 + c_2 \pi = 0 \end{array} \right\} \Rightarrow c_1 = c_2 = 0$$

iii) If $\lambda > 0$, write λ as $\lambda = \alpha^2$.

Then the general solution is

$$y = C_1 \cos(\alpha x) + C_2 \sin(\alpha x)$$

The boundary condition yield

$$\left.\begin{cases} C_1 \cos 0 + C_2 \sin 0 = 0 & \Rightarrow C_1 = 0 \\ C_1 \cos(\alpha\pi) + C_2 \sin(\alpha\pi) = 0 & \end{cases}\right\} \Rightarrow C_2 \sin(\alpha\pi) = 0$$

To allow a non-trivial solution, we require $C_2 \neq 0$, that means.

$\sin(\alpha\pi) = 0 \Rightarrow \alpha\pi = k\pi$, k is integer.

$\alpha = k$.

Then $\lambda = k^2$, $k = 1, \dots n, \dots$

we call such a λ ~~the~~ an eigenvale of the endpoint problem.

the corresponding solution $y = \sin(k\pi)$ eigenfunction.

(choose $C_2 = 1$).