Equivalence between higher order DEs and first order systems.

1. Higher-order DE $\longrightarrow$ first order system

General form $\quad y^{(n)} = f(t, y, y', \cdots, y^{(n-1)})$

eg.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t)$$

$$\Rightarrow y^{(n)} = \frac{-(a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y)}{a_n} + \frac{f(t)}{a_n}$$

set

$$x_0 = y$$
$$x_1 = y'$$
$$\vdots$$
$$x_{n-1} = y^{(n-1)}$$

then

$$x_0' = y' = x_1$$
$$x_1' = y'' = x_2$$
$$\vdots$$
$$x_{n-1}' = y^{(n)} = -\frac{a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y}{a_n}$$
$$= -\frac{a_{n-1} x_{n-1} + \cdots + a_1 x_1 + a_0 x_0}{a_n} + \frac{f(t)}{a_n}$$

$\downarrow$ write in matrix form

$$\underbrace{\begin{bmatrix} x_0' \\ x_1' \\ \vdots \\ \vdots \\ x_{n-1}' \end{bmatrix}}_{\dot{\vec{x}}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ & 0 & & & \ddots & 0 \\ -\frac{a_{n-1}}{a_n} & -\frac{a_{n-2}}{a_n} & \cdots & & \cdots & -\frac{a_0}{a_n} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ f(t) \end{bmatrix}}_{\vec{F}(t)} \Longleftrightarrow \frac{d\vec{x}}{dt} = A\vec{x} + \vec{F}(t)$$

2、 first order DE systems $\rightarrow$ higher order DEs

i) Some notation : $Dx = \frac{d}{dt}(x)$ $D^2 x = \frac{d^2}{dt^2} x$

$$(aD+b)\,x = a\frac{dx}{dt} + bx .$$

ii) Define $L_{ij} = a_{ij}D + b_{ij}$, then first order DE systems could be written as

$$L\vec{X} = \vec{F}(t)$$

$$\begin{bmatrix} a_{11}D+b_{11}, & a_{12}D+b_{12}, & \cdots, & a_{1n}D+b_{1n} \\ a_{21}D+b_{21}, & a_{22}D+b_{22}, & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1}D+b_{n1}, & - & - & a_{nn}D+b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

By Crammer's rule.

$$x_i = \frac{\begin{vmatrix} L_{11} & \cdots & f_1(t) & \cdots & L_{1n} \\ \vdots & & \vdots & & \vdots \\ L_{n1} & \cdots & f_n(t) & \cdots & L_{nn} \end{vmatrix}}{|L|} \Rightarrow |L|\,x_i = \begin{vmatrix} L_{11} & \cdots & f_1(t) & \cdots & L_{1n} \\ \vdots & & \vdots & & \vdots \\ L_{n1} & \cdots & f_n(t) & \cdots & L_{nn} \end{vmatrix}$$

If $\vec{F(t)} = 0$, then x_i satisfies $|L|\,x_i = 0$.

Example 1. $t^2x'' + tx' + (t^2-1)x = 0$ ... ①

transform the differential equation into a first order DEs system

we define $x_1 = x$, $x_2 = x' = x_1'$, then substitute into ①

$$t^2 x_2' + t x_2 + (t^2-1)x_1 = 0 \Rightarrow x_2' = -\frac{t x_2 + (t^2-1)x_1}{t^2}$$

Hence we have
$$\begin{cases} x_1' = x_2 \\ x_2' = -\dfrac{t x_2 + (t^2-1)x_1}{t^2} \end{cases}$$

Example 2.

Transform the system

$$\begin{cases} (D-4)x + 3y = 0 \\ -6x + (D+7)y = 0 \end{cases}$$ into a second order DE.

$$L = \begin{pmatrix} D-4 & 3 \\ -6 & D+7 \end{pmatrix}$$

$$|L| = (D-4)(D+7) + 18 = D^2 + 3D - 10$$

Then x satisfies

$$(D^2 + 3D - 10)x = 0$$

that is $$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - 10x = 0$$

we can solve it by the knowledge of chapter 2.

Some conception.

i) the system $L\vec{x} = \vec{F}(t)$ is degenerate if $|L| = 0$.

ii) A degenerate system may either have no solution or infinitely many solution.

Example 3

$$\begin{cases} (D+2)x + (D+2)y = t \\ (D+3)x + (D+3)y = t^2 \end{cases}$$

$$L = \begin{pmatrix} D+2 & D+2 \\ D+3 & D+3 \end{pmatrix} \qquad |L| = (D+2)(D+3) - (D+3)(D+2) = 0$$

Thus the system is degenerate.

~~x satisfies~~ The system has infinitely many solution if $\begin{vmatrix} D+2 & t \\ D+3 & t^2 \end{vmatrix} = 0$

has no solution if $\begin{vmatrix} D+2 & t \\ D+3 & t^2 \end{vmatrix} \neq 0$

However $\begin{vmatrix} D+2 , & t \\ D+3 , & t^2 \end{vmatrix} = (D+2)t^2 - (D+3)t = 2t + 2t^2 \cancel{+1+3t} \cancel{\neq 0}$

$$-(1+3t)$$

$$= 2t^2 - t - 1 \neq 0$$

Hence The system has no solution.

3. The general solution of $\frac{d\vec{X}}{dt} = p(t)\vec{X} + \vec{F}(t)$

is $\vec{X}(t) = \vec{C_1}\vec{X_1}(t) + C_2\vec{X_2}(t) + \cdots + C_n\vec{X_n}(t) + \vec{X_p}(t)$

where $x_1(t), \cdots, x_n(t)$ are n independent solution of homogenous linear equation

$$\frac{d\vec{X}(t)}{dt} = p(t)\vec{X}$$

Remark $x_i(t)$ are linear independent if $W(\vec{x_1}, \vec{x_2}, \cdots, \vec{x_n}) \neq 0$.

$$W = \begin{vmatrix} x_{11}(t), & x_{12}(t), & \cdots & , x_{1n}(t) \\ x_{21}(t), & x_{22}(t), & \cdots & , x_{2n}(t) \\ \vdots & & & \\ x_{n1}(t), & x_{n2}(t), & \cdots & , x_{nn}(t) \end{vmatrix}$$

Example 4.

$$x_1(t) = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}, \quad x_2(t) = \begin{bmatrix} -2e^{3t} \\ 0 \\ e^{-3t} \end{bmatrix}, \quad x_3(t) = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}$$

are solution of the equation $\frac{d\vec{x}}{dt} = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & 2 \\ 0 & -1 & 3 \end{bmatrix} \vec{x}$.

Are they linear independent?

$$W(x_1, x_2, x_3) = \begin{vmatrix} 2e^t & 2e^{3t} & 2e^{5t} \\ 2e^t & 0 & -2e^{5t} \\ e^t & e^{3t} & e^{5t} \end{vmatrix} = -16e^{9t} \neq 0,$$

Hence they are independent.