The Eigenvalue method for Homogeneous system.

we solve homogenous. first-order linear system with constant coefficient.

$$\frac{d\vec{x}}{dt} = A\vec{x} \quad \cdots \quad ②$$

we try to find n independent solutions

Try the form of $\vec{x} = \vec{v} e^{\lambda t}$

then $\frac{d\vec{x}}{dt} = \lambda \vec{v} e^{\lambda t}$ ... Substitute into ②

$$\lambda \vec{v} e^{\lambda t} = A \vec{v} e^{\lambda t} \Rightarrow A\vec{v} = \lambda \vec{v}$$

← eigenvector of A

eigenvalue of A

λ satisfies

$$|A - \lambda I| = a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0$$

a polynomial

it ~~might have~~ has n roots.

Then we can find associated eigenvectors

see Page 367-369 for a detailed description of The eigenvalue method.

b. Three Cases

1. Distinct real eigenvalues. $\lambda = \lambda_1, \cdots, \lambda_n$.

Then the general solution is

$$x = c_1 \vec{v_1} e^{\lambda_1 t} + c_2 \vec{v_2} e^{\lambda_2 t} + \cdots + c_n \vec{v_n} e^{\lambda_n t}$$

See P_{369} Example 1.

2. Complex eigenvalues. $\lambda = p + qi$, $\vec{v}$ = complex-valued vector.

$x_1(t) = Re(\vec{v} e^{\lambda t})$

$x_2(t) = Im(\vec{v} e^{\lambda t})$

$i^2 = -1$

$e^{i n\theta} = \cos n\theta + i \sin n\theta$

See P_{374} and P_{375} Example 3

3. repeated roots. we discuss the case where A is a 3x3 matrix in detail, the higher dimension case ~~can't be~~ is quite complicated, and you can refer to P_{401}.

If A is a 3X3 matrix, then the repeated roots may have multiplicity 2 or 3.

i) the multiplicity is 2 and we can find 2 independent eigenvectors v_1, v_2, by solving $(A-\lambda_1 I)V=0$ associated with λ_1. then the general solution is

$$x = c_1\vec{V_1}e^{\lambda_1 t}+c_2\vec{V_2}e^{\lambda_1 t} + c_3\vec{V_3}e^{\lambda_2 t}.$$

ii) ~~the~~ multiplicity 2, only 1 independent eigenvector v_1, then

$$x = c_1(\vec{V_1}t+\vec{V_2})e^{\lambda_1 t} + c_2\vec{V_1}e^{\lambda_1 t} + c_3\vec{V_3}e^{\lambda_2 t}$$

$\vec{V_2}$ satisfies $(A-\lambda_1 I)^2 V_2=0$

$\vec{V_1}$ satisfies $V_1=(A-\lambda_1 I)V_2$.

we call the eigenvalue complete.

iii) multiplicity 3, 3 independent eigenvector v_1, v_2, v_3 by solving $(A-\lambda_1 I)=0$

we

$$x = e^{\lambda_1 t}(c_1\vec{V_1}+c_2\vec{V_2}+c_3\vec{V_3})$$

iv) multiplicity 3, 2 independent eigenvector $\vec{V_1}, \vec{V_3}$

$$x=c_1e^{\lambda_1 t}\vec{V_1} + c_2(\vec{V_1}t+\vec{V_2})e^{\lambda_1 t} + c_3\vec{V_3}e^{\lambda_1 t}.$$

$\vec{V_2}$ satisfies $(A-\lambda_1 I)^2\vec{V_2}=0$

$\vec{V_1}$ satisfies $\vec{V_1} = (A-\lambda_1 I)\vec{V_2}$

v) multiplicity 3 only 1 independent vector by solving $(A\lambda - I)V = 0$

then

$$x = c_1 e^{\lambda_1 t}\vec{V_1} + c_2(\vec{V_1}t + \vec{V_2})e^{\lambda_1 t} + c_3\left(\frac{\vec{V_1}}{2}t^2 + \vec{V_2}t + \vec{V_3}\right)e^{\lambda_1 t}$$

V_3 satisfies: $(A-\lambda I)^3 V_3 = 0$

V_2 ~~sa~~ $V_2 = (A-\lambda I)V_3$

$V_1 = (A-\lambda I)^2 V_3$

Example for iv) and v)

iv)

$$A = \begin{bmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

$$|A-\lambda I| = (1-\lambda)^3$$

$\lambda = 1$ has multiplicity 3.

2 independent vector.

$$(A-\lambda I)\vec{V} = 0 \Rightarrow \vec{V} = k_1\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + k_2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \cdots ③$$

So the general solution is

$$\vec{x} = c_1 e^{-t}\vec{V_1} + c_2(\vec{V_1}t + V_2)e^{-t} + c_3 e^{-t}\vec{V_3}$$

first, we solve $(A-I)^2 V_2 = 0$,

$$(A-I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ Thus any non-zero vector V_2 is a solution.

we choose $V_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

then $V_1 = (A-I)V_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

any V_3 satisfies 3 and independent of V_1 is enough, we choose $V_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Hence the general solution is

$$\vec{x} = c_1 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} e^{+t} + c_2 \begin{bmatrix} -3t+1 \\ t \\ t \end{bmatrix} e^{+t} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{+t}$$

2)

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}$$

$|A - \lambda I| = -(1+\lambda)^3$

$\Rightarrow \lambda = -1$ with multiplicity 3

$$(A+I) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{bmatrix}$$

$(A+I)V = 0$

$\Rightarrow V = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ← 1 independent vector.

So the general solution is

$$\vec{x} = C_1 e^{-t}\vec{V_1} + C_2(\vec{V_1}t + V_2)e^{-t} + C_3\left(\frac{V_1}{2}t^2 + V_2 t + V_3\right)e^{-t}$$

~~First~~ ~~we~~

$$(A+I)^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(A+I)^3 V_3 = 0 \implies \text{any nonzero vector is a solution}$$

we choose $V_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

then $V_2 = (A+I)V_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

$V_1 = (A+I)V_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Hence

$$\vec{x} = C_1 e^{-t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_2 e^{-t}\begin{bmatrix} t \\ 2 \\ 1 \end{bmatrix} + C_3 e^{-t}\begin{bmatrix} \frac{t^2}{2} \\ 2t+1 \\ t \end{bmatrix}$$

If you need more examples

See P399 Example 4.