Matrix Exponentials

$\vec{x}_1(t)$, $\vec{x}_2(t)$, ..., $\vec{x}_n(t)$ are n independent solutions of

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

Define $n \times n$ matrix.

fundamental matrix ← $\Phi(t) = [\, x_1(t), x_2(t), \ldots, x_n(t)]$

Then the general solution could be written as

$$\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + \cdots + c_n \vec{x}_n(t) = \Phi(t) \cdot \vec{c}$$

If we have initial condition $\vec{x}(0) = \vec{x}_0$, then $\Phi(0) \cdot \vec{c} = \vec{x}_0 \Rightarrow \vec{c} = \Phi(0)^{-1} \vec{x}_0$

Thus the general solution is given by

$$\vec{x}(t) = \Phi(t) \Phi(0)^{-1} x_0 \quad \cdots \quad ①$$

Example 1. find a fundamental matrix for the system and apply ① to find a solution satisfying the given initial condition

$$\vec{x}' = \begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix} \vec{x} \qquad \vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix} \qquad |A - \lambda I| = (2-\lambda)(-2-\lambda) + 20 = \lambda^2 + 16$$

$$|A - \lambda I| = 0 \Rightarrow \lambda = \pm 4i$$

$$(A - 4iI)\, v = 0 \Rightarrow \begin{bmatrix} 2-4i & -5 \\ 4 & -2-4i \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0 \Rightarrow \vec{v} = k_2 \begin{bmatrix} \frac{5}{2-4i} \\ 1 \end{bmatrix}$$

choose $\vec{v} = \begin{bmatrix} \frac{5}{2-4i} \\ 1 \end{bmatrix}$

$$\vec{v} \cdot e^{4it} = \begin{bmatrix} \frac{5}{2-4i} \\ 1 \end{bmatrix} \cdot (\cos 4t + i \sin 4t) = \begin{bmatrix} \frac{\cos 4t - 2\sin 4t}{2} \\ \cos 4t \end{bmatrix} + i \begin{bmatrix} \cos 4t + \frac{\sin 4t}{2} \\ \sin 4t \end{bmatrix}$$

Thus two independent solution are

$$\begin{bmatrix} \frac{\cos 4t - 2\sin 4t}{2} \\ \cos 4t \end{bmatrix} \cdot \begin{bmatrix} \cos 4t + \frac{\sin 4t}{2} \\ \sin 4t \end{bmatrix}$$

the fundamental matrix

$$\Phi = \begin{bmatrix} \frac{\cos 4t - 2\sin 4t}{2} & \cos 4t + \frac{\sin 4t}{2} \\ \cos 4t & \sin 4t \end{bmatrix}$$

$$\Phi(0) = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Phi^{-1}(0) = -1 \cdot \begin{bmatrix} 0 & -1 \\ -1 & \frac{1}{2} \end{bmatrix}, \quad \Phi^{-1}(0)\, \vec{x}_0 = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$$

Thus the general solution is

$$\vec{X}(t) = \begin{bmatrix} -\frac{5}{4} \sin 4t \\ \cos 4t - \frac{\sin 4t}{2} \end{bmatrix}$$

Exponential Matrices.

Define
$$e^{A} = I + A + \frac{A^2}{2!} + \cdots + \frac{A^n}{n!} + \cdots$$

then

$$\frac{d}{dt}\left(e^{At}\right) = \frac{d}{dt}\left(I\cancel{t} + At + \frac{(At)^2}{2!} + \cdots + \frac{A^n}{n!} \right)$$

$$= A + A^2 t + \frac{A^3 t^2}{2!} + \cdots$$

$$= A\left(I + At + A^2\frac{t^2}{2!} + \cdots \right)$$

$$= A e^{At}$$

Hence the solution of the initial value problem $\vec{X}' = A\vec{X}$, $\vec{x}(0) = \vec{x}_0$

is given by $x(t) = e^{At}\vec{X}_0$.

compared with ①, we know that.

$$e^{At} = \Phi(t)\Phi^{-1}(0)$$

How to compute e^{A}

example 2. $A = \begin{bmatrix} 6 & -6 \\ 4 & -4 \end{bmatrix}$

$$\det(A - \lambda I) = (6-\lambda)(-4-\lambda) + 24 = \lambda(\lambda-2) = 0 \Rightarrow \lambda = 0, 2$$

$(A-0I)v \Rightarrow v = k_1\begin{bmatrix}1\\1\end{bmatrix}$ choose $\vec{V_1} = \begin{bmatrix}1\\1\end{bmatrix} \Rightarrow x_1(t) = V_1 e^{0t} = \begin{bmatrix}1\\1\end{bmatrix}$

$(A-2I)v \Rightarrow v = k_2\begin{bmatrix}3\\2\end{bmatrix}$ choose $\vec{V_2} = \begin{bmatrix}3\\2\end{bmatrix} \Rightarrow x_2(t) = V_2 e^{2t} = \begin{bmatrix}3e^{2t}\\2e^{2t}\end{bmatrix}$

Hence $\Phi(t) = \begin{pmatrix}1 & 3e^{2t}\\1 & 2e^{2t}\end{pmatrix}$ $\Phi(0) = \begin{pmatrix}1 & 3\\1 & 2\end{pmatrix}$, $\Phi^{-1}(0) = -1\begin{pmatrix}2 & -3\\-1 & 1\end{pmatrix}$

$$e^{At} = \Phi(t)\,\Phi(0)^{-1} = \begin{pmatrix}3e^{2t}-2 & 3-3e^{2t}\\-2+2e^{2t} & 3-2e^{2t}\end{pmatrix}$$

show that the matrix is nilpotent and then use this fact to find e^{At}

Example 3. $A = \begin{bmatrix}6 & 4\\-9 & -6\end{bmatrix}$

$$A^2 = \begin{bmatrix}6 & 4\\-9 & -6\end{bmatrix}\begin{bmatrix}6 & 4\\-9 & -6\end{bmatrix} = 0$$

Thus $e^{At} = I + At + \frac{1}{2}A^2t^2 + \cdots$

$= I + At + 0 + \cdots$

$= \begin{bmatrix}1+6t & 4\\-9 & 1-6t\end{bmatrix}$

Nonhomogeous Linear system

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}(t)$$

general solution

$$\vec{x}(t) = \vec{x}_c(t) + \vec{x}_p(t)$$

1. Undetermined coefficients (f(t) must be a linear combination of products of polynomials, exponential functions and sines and cosines)

step 1: find the complemetary function (homogenous solution)

step 2: make a guess as to the general form of a particular solution.

step 3: determine the coefficients of x_p by substitution.

Example 1.

$$x' = x - 2y$$

$$y' = 2x - y + e^t \sin t$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}}_{\leftarrow A} \begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ e^t \sin t \end{bmatrix}}_{\leftarrow f(t)} \quad \cdots ①$$

step 1. $\det(A - \lambda I) = (1-\lambda)(-1-\lambda) + 4 = \lambda^2 + 3$

$$\begin{bmatrix} x_c \\ y_c \end{bmatrix} = c_1 \begin{bmatrix} 2\cos\sqrt{3}t \\ \cos\sqrt{3}t + \sqrt{3}\sin\sqrt{3}t \end{bmatrix} + c_2 \begin{bmatrix} 2\sin\sqrt{3}t \\ -\sqrt{3}\cos\sqrt{3}t + \sin\sqrt{3}t \end{bmatrix}$$

Step 2

Based on $f(t)$ and $\vec{x_c}$, it is reasonable to select a trial particular solution.

$$x_p = \vec{a} e^t \sin t + \vec{b} e^t \cos t$$

then $$x_p' = \vec{a} (e^t \sin t + e^t \cos t) + \vec{b} e^t (\cos t - \sin t)$$

Substitute into ①, we have

$$\vec{a} e^t (\sin t + \cos t) + \vec{b} e^t (\cos t - \sin t) = A(\vec{a} e^t \sin t + \vec{b} e^t \cos t) + \begin{pmatrix} 0 \\ e^t \sin t \end{pmatrix}$$

$$\begin{bmatrix} 0 \\ e^t \sin t \end{bmatrix} + \begin{bmatrix} a_1 e^t (\sin t + \cos t) + b_1 e^t (\cos t - \sin t) \\ a_2 e^t (\sin t + \cos t) + b_2 e^t (\cos t - \sin t) \end{bmatrix} = \begin{bmatrix} (a_1 - 2a_2) e^t \sin t + (b_1 - 2b_2) e^t \cos t \\ (2a_1 - a_2) e^t \sin t + (2b_1 - b_2) e^t \cos t \end{bmatrix}$$

$$\left\{ \begin{array}{l} a_1 - b_1 = a_1 - 2a_2 \Rightarrow b_1 = 2a_2 \\ a_1 + b_1 = b_1 - 2b_2 \Rightarrow a_1 = 2b_2 \\ a_2 - b_2 = 2a_1 - a_2 + 1 \\ a_2 + b_2 = 2b_1 - b_2 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} a_1 = -\frac{6}{13} \\ b_1 = \frac{4}{13} \\ a_2 = \frac{2}{13} \\ b_2 = \frac{3}{13} \end{array} \right.$$

2、 Variation of parameter (works for any $\vec{f}$)

$$\vec{x}_p(t) = \Phi(t)\,\vec{u}(t)$$

$$\left.\begin{aligned} \frac{d\vec{x}_p}{dt} &= \frac{d\Phi}{dt}\cdot\vec{u}(t) + \Phi\frac{d\vec{u}}{dt} \\ A\vec{x}_p + \vec{f}(t) &= A\Phi(t)\vec{u}(t) + \vec{f}(t) \\ \frac{d\Phi}{dt} &= A\Phi(t)\,\cancel{\vec{u}(t)} \end{aligned}\right\} \Rightarrow \Phi\frac{d\vec{u}}{dt} = \vec{f}(t) \Rightarrow \frac{d\vec{u}}{dt} = \Phi^{-1}f$$

$$\Rightarrow u = \int \Phi^{-1} f\, dt$$

Hence $\quad \vec{x}_p(t) = \Phi(t)\int \Phi^{-1} f\, dt$

The general solution. If initial condition is $x(a) = x_a$

$$x(t) = \Phi(t)\Phi(a)^{-1}x_a + \Phi(t)\int_a^t \Phi(s)^{-1} f(s)\, ds \quad \cdots ③$$

$$e^{A(t-a)} = \Phi(t)\Phi(a)^{-1}$$

$$\Phi(t) = e^{A(t-a)}\Phi(a) \quad \cdots ①$$

$$\Phi(t)^{-1} = \Phi^{-1}(a)\, e^{-A(t-a)} \quad \cdots ②$$

Plug ① ② into ③

$$x(t) = e^{A(t-a)}x_a + e^{At}\int_a^t e^{-As} f(s)\, ds \quad \cdots ④$$

Example Solve the initial value problem $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}$

~~where~~ given $A = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}$ $\quad f(t) = \begin{bmatrix} 0 \\ t^{-2} \end{bmatrix}$ $\quad x(1) = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ $\quad e^{At} = \begin{bmatrix} 1+3t & -t \\ 9t & 1-3t \end{bmatrix}$

we use ④

$$e^{-At} f(t) = \begin{bmatrix} 1-3t\ , & t \\ -9t\ , & 1+3t \end{bmatrix} \begin{bmatrix} 0 \\ t^{-2} \end{bmatrix} = \begin{bmatrix} t^{-1} \\ \frac{1+3t}{t^2} \end{bmatrix}$$

$$\Rightarrow \int_1^t e^{-As} f(s)\, ds = \cancel{\begin{bmatrix} \ln t \\ -\frac{1}{t}+3\ln t \end{bmatrix}} \begin{bmatrix} \ln t \\ -\frac{1}{t}+3\ln t+1 \end{bmatrix}$$

$$\Rightarrow e^{At} \int_1^t e^{-A(s)} f(s)\, ds = \begin{bmatrix} 1+3t & -t \\ 9t & 1-3t \end{bmatrix} \cdot \begin{bmatrix} \ln t \\ -\frac{1}{t}+3\ln t+1 \end{bmatrix} = \begin{bmatrix} 1+\ln t-t \\ -\frac{1}{t}+4+3\ln t-3t \end{bmatrix}$$

$$e^{A(t-1)} = \begin{bmatrix} 1+3(t-1) & -(t-1) \\ 9(t-1) & 1-3(t-1) \end{bmatrix} = \begin{bmatrix} 3t-2 & 1-t \\ 9t-9 & 4-3t \end{bmatrix}$$

$$\Rightarrow e^{A(t-1)} \cdot x(1) = \begin{bmatrix} 3t-2 & 1-t \\ 9t-9 & 4-3t \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 2t+1 \\ 6t+1 \end{bmatrix}$$

Thus

$$x(t) = e^{A(t-1)} \cdot x(1) + e^{At} \int_1^t e^{-A(s)} f(s)\, ds = \begin{bmatrix} 2+\ln t+t \\ 5+3t-\frac{1}{t}+3\ln t \end{bmatrix}$$