First-ORDER ODEs

Standard form: $y' = f(x,y)$

I. Geometric view of ODEs.

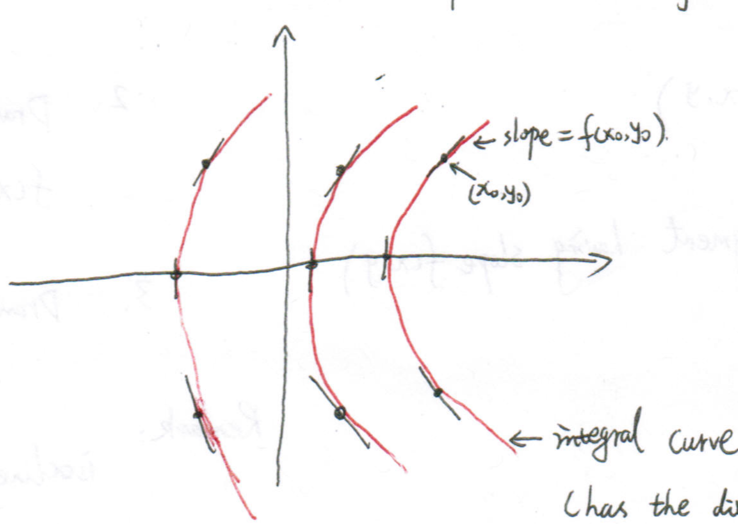
Analytic: Geometric

$y' = f(x,y)$ $\longleftrightarrow$ Direction field

Find the analytic solution $y_1(x)$
Such that $y_1'(x) = f(x, y_1(x))$ $\longleftrightarrow$ integral curve

Direction field [small line segment with slope $= f(x_0, y_0)$ at the point (x_0, y_0)]

Integral curve [A curve go through the plane and at every point is ~~ta~~ tangent to the small line segment]



(has the direction of the field everywhere)

Remark: Writing down the differential equation is the same thing as drawing the direction field.

Solving analytically for a solution is the same thing as drawing the integral curve geometrically.

$y_1(x)$ solution to $y' = f(x,y) \iff$ the graph of $y_1(x)$ is an integral curve.

How to draw direction field ?

Computer	Human
1. pick the point (x,y) as many as you want.	1. pick the slope C
2. Compute $f(x,y)$	2. Draw the curve (isocli $f(x,y) = C$
3. draw line segment having slope $f(x,y)$	3. Draw line segment

Remark: isocline contains all the p

in the plane where the ~~to~~ slope of line segme

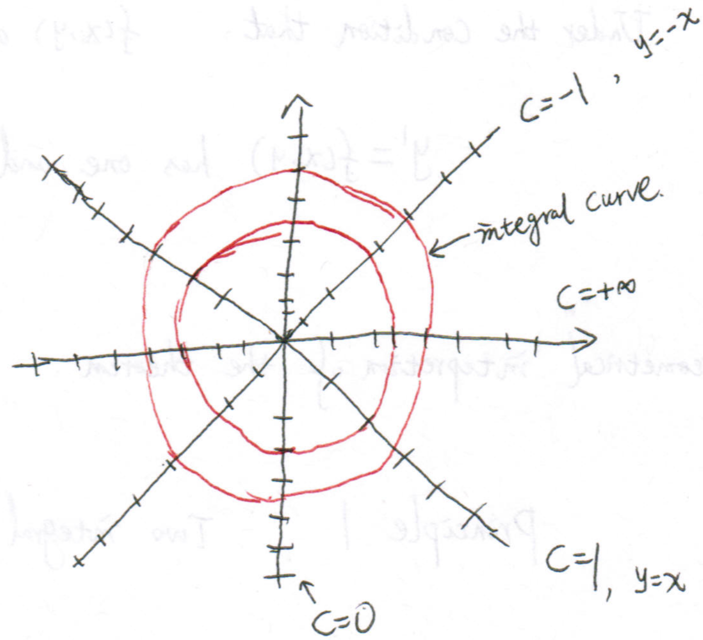
would be C.

Example 1

$$y' = -\frac{x}{y}$$

isoclines

$$\frac{-x}{y} = C \iff y = -\frac{1}{C}x$$



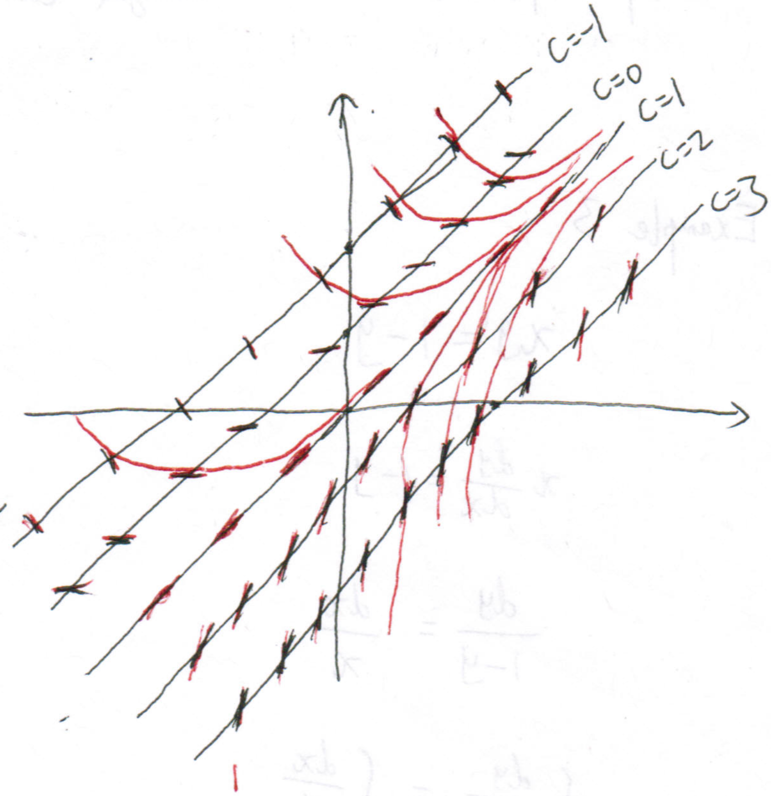
Example 2

$$\cancel{y' = -\frac{1}{c}x}$$

$$y' = 1 + x - y$$

isoclines

$$1 + x - y = C \iff y = x + 1 - C$$



II Solving the equation analytically.

Separation of variable:

$$\frac{dy}{dx} = g(x) \cdot h(y)$$

separating the variable: $\quad \dfrac{dy}{h(y)} = g(x)\,dx$

ntegrating both side $\quad \displaystyle\int \frac{dy}{h(y)} = \int g(x)\,dx$

Example 4.

$$\frac{dy}{dx} = \frac{x^2}{y}, \quad y(0) = 1$$

$$y\,dy = x^2\,dx$$

$$\int y\,dy = \int x^2\,dx$$

$$\frac{y^2}{2} = \frac{x^3}{3} + C$$

To find C, use the initical condition.

$$\underset{\frac{1}{2}}{\underset{\parallel}{\frac{y^2(0)}{2}}} = \frac{0^3}{3} + C \Rightarrow C = \frac{1}{2}$$

Uniquess and Existence Theorem.

Under the condition that: $f(x,y)$ and $f_y(x,y)$ is continuous near (x_0,y_0).

$y' = f(x,y)$ has one and only one solution through the point (x_0,y_0)

Geometrical intepretion of the theorem.

Principle 1. Two integral curve can not cross.

Principle 2. Two integral curve can not touch (tangent)

Example 3

$$xy' = 1-y$$

$$x\frac{dy}{dx} = 1-y$$

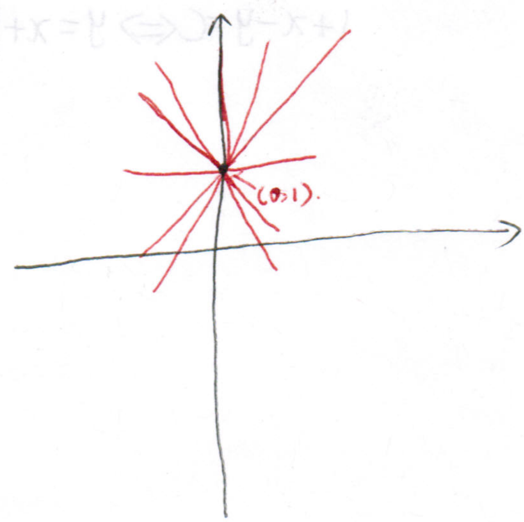
$$\frac{dy}{1-y} = \frac{dx}{x}$$

$$\int \frac{dy}{1-y} = \int \frac{dx}{x}$$

$$-\ln(1-y) = \ln x + C.$$

$$\Leftrightarrow \quad 1-y = C_1 x$$

Draw the solution from the analytical solution for all values of C.



No uniqueness at the point (0,1).

Is it a violation of the theorm?

No, $f(x,y) = \frac{1-y}{x}$, which is discontinuous at $x=0$, Thus the theorem can not gurantee the uniqueness.

First order linear Equation.

Standard linear form: $y' + p(x)y = q(x)$.

what makes ~~it~~ u a integrating factor is that, after I multiple the DE by u, I want it turn out to be $(uy)'$ in the left hand side.

$$\text{LHS} = uy' + upy \overset{?}{=} (uy)' = uy' + u'y$$

it works if $u' = up \xrightarrow{\text{solve it by Separation of variable}} u = e^{\int p(x)\,dx}$.

$$\text{RHS} = u(x)\,q(x)$$

Method:

$$y' + p(x)\,y = q(x)$$

① Transform it into standard linear form

② Calculate integrating factor $e^{\int p(x)\,dx}$

③ multiply ① by $e^{\int p(x)\,dx}$

Example 5 $\quad xy' - y = x^3$

① $$y' - \frac{y}{x} = x^2$$

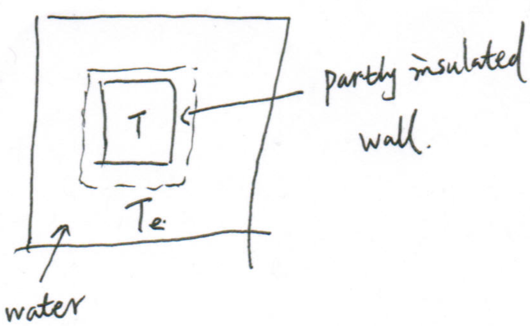
② $$e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

③ $$\frac{1}{x}\left(y' - \frac{y}{x}\right) = x^2 \cdot \frac{1}{x} \Leftrightarrow \frac{1}{x}y' - \frac{1}{x^2}y = x \Rightarrow \left(\frac{1}{x}y\right)' = x$$

integrating. $$\frac{1}{x}y = \int x\,dx \Leftrightarrow \frac{1}{x}y = \frac{x^2}{2} + C \Leftrightarrow y = \frac{x^3}{2} + C$$

Model

Temp-diffusion.



What is diffential equation to model this situation

(Newton's cooling law)

$$\frac{dT}{dt} = k(T_e - T)$$

T is Temperture, $\quad t$ is time

Standard linear form:

$$\frac{dT}{dt} + kT = kT_e$$

usually constant $\qquad$ change with time.