The sad fact is that, those we have learned in last class are the only two general method there are, those are the two kinds of equations they can always be solved. To some extent, all the other equations that can be solved is the kind of equation that can be transformed into one of those two by substitution.

Object: Find good substitution to reduce the problem.

a) $\dfrac{dy}{dx} = F(ax+by+c)$, $v = ax+by+c$.

$$\Downarrow$$

$$\frac{dv}{dx} = a + b\frac{dy}{dx}$$

Plug in

$$\frac{dy}{dx} = \left(\frac{dv}{dx} - a\right)\cdot\frac{1}{b}$$

$$\left(\frac{dv}{dx} - a\right)\cdot b = F(v).$$

Then separating variable.

Example 1 $\qquad y' = (x+y+1)^2 \quad \cdots ①$

Try $v = x+y+1$, then $\dfrac{dv}{dx} = 1 + \dfrac{dy}{dx}$, plug back in ①

we have $\dfrac{dv}{dx} - 1 = v^2$

$$\frac{dv}{dx} = 1+v^2 \xrightarrow{\text{separating variable}} \frac{dv}{1+v^2} = dx \xrightarrow{\text{integrating}} \arctan v = x+C$$

$$\Rightarrow \ v = \tan(x+C) \xrightarrow{\text{change back to } y} \cancel{x = \tan(x+y+1)} \ \ x+y+1 = \tan(x+C)$$

b) Homogenous Equation.

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

make substitution $v = \frac{y}{x}$

$\Leftrightarrow \ y = x \cdot v$

plug in

$$\frac{dy}{dx} = v + \frac{dv}{dx}$$

$$v + \frac{dv}{dx} = F(v)$$

separating variable.

Example 2

$$xyy' = x^2 + 3y^2$$

$$y' = \frac{x^2+3y^2}{xy} = \frac{x}{y} + \frac{3y}{x}, \quad \text{Let } v = \frac{y}{x}$$

$$v + \frac{dv}{dx} = \frac{1}{v} + 3v$$

$$\frac{dv}{dx} = \frac{1+2v^2}{v}$$

$$\frac{v}{1+2v^2}\, dv = dx$$

$$\frac{1}{4}\ln(1+2v^2) = x+C \iff \frac{1}{4}\ln\left(1+2\frac{y^2}{x^2}\right) = x+C$$

c) Bernoulli Equation.

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$$

Looks similar to the first-order linear ODE.

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{P(x)}{y^{n-1}} = Q(x) \quad , \text{ try } \quad v = \frac{1}{y^{n-1}}$$

$$\frac{dv}{dx} = -(n-1)\cdot\frac{1}{y^n}\cdot\frac{dy}{dx}$$

$$\frac{1}{1-n}\frac{dv}{dx} + P(x)\,v = Q(x)$$

$$\underbrace{\frac{dv}{dx} + (1-n)P(x)\,v = (1-n)Q(x)}$$

First order linear ODE.

Example 3: $y^2y' + 2xy^3 = 6x$

$$\frac{dy}{dx} + 2xy = 6xy^{-2} \qquad n=-2. \qquad , \text{ try } \quad v = \frac{1}{y^{-2-1}} = y^3$$

$$y^2\frac{dy}{dx} + 2xy^3 = 6x \qquad\qquad \frac{dv}{dx} = 3y^2\cdot\frac{dy}{dx}$$

$$\frac{1}{3}\frac{dv}{dx} + 2xv = 6x$$

Solve it we get $v = 3 + ce^{-3x^2}$

Exact Differential Equations

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \cdots ①$$

if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, then we can find a function F, s.t. $\dfrac{\partial F}{\partial x} = M$, $\dfrac{\partial F}{\partial y} = N$

Then ① is exact., the solution to ① is $F(x,y) = C$.

How to find F?

$$F = \int M(x,y)\,d\cancel{y}x + g(y)$$

$$g(y) = \int \left(N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx\right) dy$$

Example 4

$$(4x-y)\,\underset{}{\overset{dx}{d\cancel{y}}} + (6y-x)\,d\cancel{x} = 0$$

$M = 4x-y \qquad \dfrac{\partial M}{\partial y} = -1$

$N = 6y-x \qquad \dfrac{\partial N}{\partial x} = -1$

$\Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Thus the equation is exact

integrating $\dfrac{\partial F}{\partial x} = M \Rightarrow F = \int (4x-y)\,dx + g(y) = 2x^2 - xy + g(y)$

plug into $\dfrac{\partial F}{\partial y} = N \Rightarrow -x + \dfrac{\partial g}{\partial y} = 6y - x \Rightarrow \dfrac{\partial g}{\partial y} = 6y \Rightarrow g = 3y^2 + C_1$

$F = 2x^2 - xy + 3y^2 + C_1$, so the solution is $2x^2 - xy + 3y^2 = C$

Reducible Second-order Equations.

(i) $y'' = f(x, y')$

Let $v = y'$, Then $v' = f(x, v)$ ← first order

Example 5 $xy'' + y' = 4x \Rightarrow y'' = -\frac{y'}{x} + 4x$

Let $v = y'$ then $v' = -\frac{v}{x} + 4x$ ← first order linear ODE

① $v' + \frac{v}{x} = 4x$

② integrating factor $= e^{\int \frac{1}{x} dx} = x$

③ Mutiple both ① by x, $xv' + v = 4x$

$\Rightarrow (xv)' = 4x \Rightarrow xv = 2x^2 + C_1 \Rightarrow v = 2x + \frac{C_1}{x}$

Convert back to y,

$y' = 2x + \frac{C_1}{x} \Rightarrow y = x^2 + C_1 \ln x + C_2$

~~ii)~~

ii) $y'' = g(y, y')$

$v = y'$

$$y'' = \frac{dv}{dx} = \frac{dv}{dy} \cdot \frac{dy}{dx} = \frac{dv}{dy} \cdot v$$

the ODE becomes $\frac{dv}{dy} \cdot v = g(y, v)$

Example 6:

$$y y'' = 3(y')^2 \implies y'' = \frac{3(y')^2}{y}$$

$v = y'$

then $y'' = \frac{dv}{dy} \cdot v$

plug back into ODE, we have $\frac{dv}{dy} \cdot v = \frac{3v^2}{y}$

Separating variable $$\frac{dv}{3v} = \frac{dy}{y}$$

integrating $$\ln|3v| = \ln y + C \implies v = y^3 \cdot C_1 = C_1 y^3$$

Thus $$y' = \cancel{y^3 \cdot C_1} \implies \cancel{y = \frac{y^2}{6} + C_1 y + C_2}$$

$C_1 y^3$

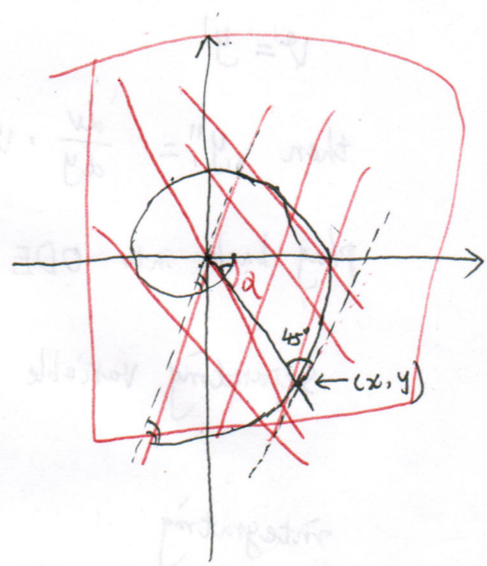
$$\frac{dy}{y^3} = C_1 dx$$

$$-\frac{1}{2} \frac{1}{y^2} = C_1 x + C_2$$

Models

I The Logarithmic spiral.

Find the trajectory of a falcon that allow the bird to keep ~~the~~ its head straight when flying. (they need to cock their heads 45 degrees to take full advantage of their sight range)



$$y' = \tan(\alpha + 45^\circ) = \frac{\tan\alpha + \tan 45^\circ}{1 - \tan\alpha \cdot \tan 45^\circ}$$

$$\tan\alpha = \frac{y}{x} \quad , \quad \tan 45^\circ = 1$$

$$y' = \frac{\frac{y}{x} + 1}{1 - \frac{y}{x}}$$ ← Homogenous.

let $v = \frac{y}{x}$ $\qquad y' = v' + xv$

$$\frac{dv}{dx} \cancel{+} xv = \frac{v+1}{1-v} \xrightarrow{\text{separating variable}} \frac{1-v}{1+v^2}\, dv = \frac{dx}{x}$$

integrating $\qquad \tan^{-1}(v) - \frac{1}{2}\ln(1+v^2) = \ln x + C$

$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\ln\left[1 + \left(\frac{y}{x}\right)^2\right] + \ln x + C$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \ln\left(\sqrt{x^2+y^2}\right) + C \xrightarrow{\text{polar coordinate}} \theta = \ln r + C$$

Population Models

The change ΔP in the population during the time interval $[t, t+\Delta t]$ of length Δt is

$$\Delta P = \{\text{birth}\} - \{\text{death}\}$$

$$\Leftrightarrow \quad \Delta P = \beta(t)\, P(t) \cdot \Delta t - \delta(t) P(t) \cdot \Delta t$$

$$\Leftrightarrow \quad \frac{\Delta P}{dt} = \beta(t)\, P(t) - \delta(t)\, P(t)$$

Case 1. with limited food supply. $\beta(t) = [\beta_0 - \beta_1 \cdot P(t)]$, $\delta(t) = \text{Constant}$

after defining new parameter

$$\frac{dP}{dt} = P \cdot (M - P)$$

M is a constant, indicate the Carrying capacity

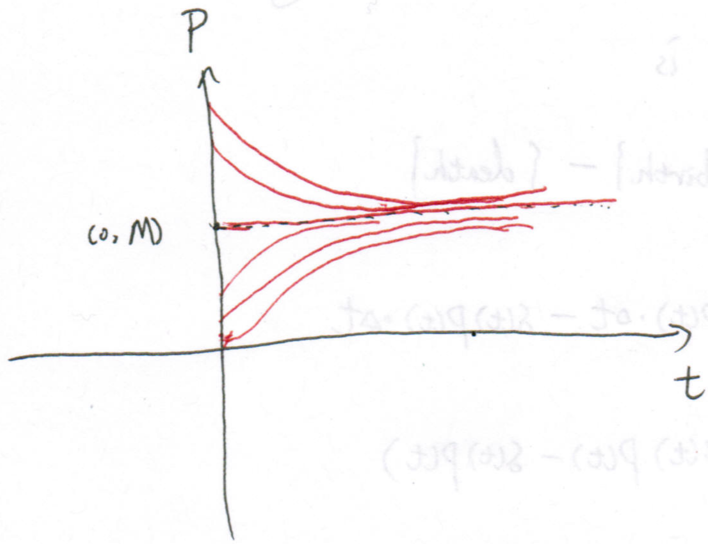
solve it using separation of variable.

Case 2 Doomsday vesus Extinction. $\beta(t) = kP$, $\delta = \text{Constant}$.

$$\frac{dP}{dt} = kP(P-M).$$

M is the threshold population

Case 1.



Case 2.