1.

a) $2xyy' = x^2 + 2y^2$ $\qquad v = \frac{y}{x} \quad vx = y \quad \therefore v'x + v = y'$

$y' = \frac{x^2+2y^2}{2xy}$

$v'x + v = \frac{1+2v^2}{2v} \qquad v'x = \frac{1+2v^2-2v^2}{2v} = \frac{1}{2v}$

$v'x = \frac{1}{2v}$

$\frac{dv}{dx} = \frac{1}{2v}\frac{1}{x}$

$2v\,dv = \frac{1}{x}dx \qquad \int 2v\,dv = \int \frac{1}{x}dx \qquad v^2 = \ln x + C$

$\therefore \left(\frac{y}{x}\right)^2 = \ln x + C_1$

$y^2 = x^2\ln x + Cx^2$

+8

b) $y^2y' + 2xy^3 = 6x \qquad y^3 = v \qquad \therefore v' = 3y^2y'$

$\therefore \frac{v'}{3} + 2xv = 6x$

$v' + 6xv = 18x$

$e^{\int 6x\,dx} = e^{3x^2}$

$v'e^{3x^2} + 6xve^{3x^2} = 18xe^{3x^2}$

$(ve^{3x^2})' = 18xe^{3x^2}$

$ve^{3x^2} = \int 18xe^{3x^2}dx$

$y^3e^{3x^2} = 3e^{3x^2} + C_1$

$y^3 = 3 + C_1e^{-3x^2}$

+8

2. $y'' + 2y' + 5y = \sin(x)$

$y'' + 2y' + 5y = 0 \qquad \therefore r^2 + 2r + 5 = 0 \qquad (r+1)^2 + 4 = 0$

+10

$r = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$

$y_1 = e^{-t}\cos 2t \qquad y_2 = e^{-t}\sin 2t$

$y_p = A\sin x + B\cos x \qquad y_p' = A\cos x - B\sin x \qquad y_p'' = -A\sin x - B\cos x$

$\therefore -A\sin x - B\cos x + 2A\cos x - 2B\sin x + 5A\sin x + 5B\cos x = \sin x$

$\begin{cases} -A - 2B + 5A = 1 \\ -B + 2A + 5B = 0 \end{cases} \Rightarrow \begin{cases} 4A - 2B = 1 \\ 4B + 2A = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{2}{10} \\ B = -\frac{1}{10} \end{cases}$

$\therefore y_p = \frac{2}{10}\sin x - \frac{1}{10}\cos x = \frac{1}{5}\sin x - \frac{1}{10}\cos x$

3. $x'' - 6x' + 8x = 2$

$\mathcal{L}\{x''\} - 6\mathcal{L}\{x'\} + 8\mathcal{L}\{x\} = \frac{2}{s}$ $\because x'(0) = x(0) = 0$

$\mathcal{L}\{x''\} = s^2\mathcal{L}\{x\}$ $\mathcal{L}\{x'\} = s\mathcal{L}\{x\}$

$s^2\mathcal{L}\{x\} - 6s\mathcal{L}\{x\} + 8\mathcal{L}\{x\} = \frac{2}{s}$

$(s^2 - 6s + 8)\mathcal{L}\{x\} = \frac{2}{s}$

$F(s) = \mathcal{L}\{x\} = \frac{2}{s(s-2)(s-4)}$

$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s(s-2)(s-4)}\right\}$

$\frac{2}{(s-2)(s-4)} = \frac{1}{s-4} - \frac{1}{s-2}$

$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\left(\frac{1}{s-4} - \frac{1}{s-2}\right)\right\}$

$= \int_0^t e^{4v} - e^{2v}\, dv$ +10

$= \left[\frac{1}{4}e^{4v}\right]_0^t - \left[\frac{1}{2}e^{2v}\right]_0^t$

$= \frac{1}{4}e^{4t} - \frac{1}{4} - \frac{1}{2}e^{2t} + \frac{1}{2}$

$x(t) = \frac{1}{4}e^{4t} - \frac{1}{2}e^{2t} + \frac{1}{4}$

4. a) $y_0 = x$

$y_1 = x' = y_0'$

$y_2 = x'' = y_1'$

$y_3 = x^{(3)} = y_2'$

$y_3' = y_4 = x^{(4)} = -y_0 + 3y_1 - 6y_2 + \sin(3t)$ +10

$$\begin{pmatrix} y_0' \\ y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 3 & -6 & 0 \end{pmatrix}\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sin(3t) \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 3 & -6 & 0 \end{pmatrix} \qquad f(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sin(3t) \end{pmatrix}$$

4

(b) $(D+2)x + (D+2)y = t$

$(2D+3)x + (D+3)y = t^2$

$\begin{vmatrix} D+2 & D+2 \\ 2D+3 & D+3 \end{vmatrix} = A$ $\quad \det A = \begin{vmatrix} D+2 & D+2 \\ 2D+3 & D+3 \end{vmatrix} = (D+2)(D+3) - (D+2)(2D+3)$

$= D^2 + 5D + 6 - 2D^2 - 4D - 3D - 6$

$= (D+2)(-D) = -D^2 - 2D$

$\begin{vmatrix} t & D+2 \\ t^2 & D+3 \end{vmatrix} = (D+3)t - (D+2)t^2$

$= 1 + 3t - 2t - 2t^2 = t - 2t^2 + 1$

$\therefore (-D^2 - 2D)x = t - 2t^2 + 1$

$\therefore -x'' - 2x' = t - 2t^2 + 1$

+10

5(a)

$x_1' = 2x_1 - 5x_2$ $\quad A = \begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix}$

$x_2' = 4x_1 - 2x_2$

$\det(A - \lambda I) = (2-\lambda)(-2-\lambda) + 20 = (\lambda+2)(\lambda-2) + 20 = \lambda^2 - 4 + 20 = \lambda^2 + 16 = 0$

$\lambda = \pm 4i$

$\lambda = 4i$: $(A - \lambda I)V = 0$

$\begin{bmatrix} 2-4i & -5 \\ 4 & -2-4i \end{bmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$4a - (2+4i)b = 0 \quad \therefore \quad \vec{V} = \begin{pmatrix} 1+2i \\ 2 \end{pmatrix}$

$\therefore \vec{V} e^{4it} = \begin{pmatrix} 1+2i \\ 2 \end{pmatrix} (\cos 4t + i \sin 4t) = \begin{pmatrix} \cos 4t - 2\sin 4t + 2i\cos 4t + i\sin 4t \\ 2\cos 4t + 2i\sin 4t \end{pmatrix}$

$\Phi(t) = \left[RE(\vec{V} e^{4it}) , Im(\vec{V} e^{4it}) \right]$

$= \begin{pmatrix} \cos 4t - 2\sin 4t , & 2\cos 4t + \sin 4t \\ 2\cos 4t , & 2\sin 4t \end{pmatrix}$

$\Phi(0) = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$ $\quad \Phi(0)^{-1} = -\frac{1}{4}\begin{pmatrix} 0 & -2 \\ -2 & 1 \end{pmatrix}$ $\quad x(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$$\Phi(0)^{-1}x(0) = -\frac{1}{4}\begin{pmatrix}0 & -2\\ -2 & 1\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix} = -\frac{1}{4}\begin{pmatrix}-6\\-1\end{pmatrix} = \begin{pmatrix}\frac{3}{2}\\ \frac{1}{4}\end{pmatrix}$$

$$\therefore \vec{x} = \Phi(t)\,\Phi(0)^{-1}x(0)$$

$$= \begin{pmatrix}\cos 4t - 2\sin 4t, & 2\cos 4t + \sin 4t\\ 2\cos 4t & 2\sin 4t\end{pmatrix}\begin{pmatrix}\frac{3}{2}\\ \frac{1}{4}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{3}{2}\cos 4t - 3\sin 4t + \frac{1}{2}\cos 4t + \frac{1}{4}\sin 4t\\ 3\cos 4t + \frac{1}{2}\sin 4t\end{pmatrix}$$

$$= \begin{pmatrix}2\cos 4t - \frac{11}{4}\sin 4t\\ 3\cos 4t + \frac{1}{2}\sin 4t\end{pmatrix}$$

+10

b) $$e^{At} = \Phi(t)\,\Phi(0)^{-1} = -\frac{1}{4}\begin{pmatrix}\cos 4t - 2\sin 4t & 2\cos 4t + \sin 4t\\ 2\cos 4t & 2\sin 4t\end{pmatrix}\begin{pmatrix}0 & -2\\ -2 & 1\end{pmatrix}$$

$$= -\frac{1}{4}\begin{pmatrix}-4\cos 4t - 2\sin 4t & -2\cos 4t + 4\sin 4t + 2\cos 4t + \sin 4t\\ -4\sin 4t & +5\,\cancel{-4}\cos 4t + 2\sin 4t\end{pmatrix}$$

$$= +\frac{1}{4}\begin{pmatrix}4\cos 4t + 2\sin 4t & -5\sin 4t\\ 4\sin 4t & 4\cos 4t - 2\sin 4t\end{pmatrix}$$

$$= \begin{pmatrix}\cos 4t + \frac{1}{2}\sin 4t & -\frac{5}{4}\sin 4t\\ \sin 4t & \cos 4t - \frac{1}{2}\sin 4t\end{pmatrix}$$

6 $$W = \begin{vmatrix}2e^{t} & 2e^{3t} & 2e^{5t}\\ 2e^{t} & 0 & -2e^{5t}\\ e^{t} & -e^{3t} & e^{5t}\end{vmatrix} = 2e^{t}(-2e^{8t}) - 2e^{3t}(2e^{6t} + 2e^{6t}) + 2e^{5t}(-2e^{4t})$$

$$= -4e^{9t} - 8e^{9t} - 4e^{9t}$$ +10

$$= -16e^{9t} \neq 0$$

The following three vectors are independent

7.

a) $$x' = \begin{bmatrix}1 & -4\\ 4 & 9\end{bmatrix}x$$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -4 \\ 4 & 9-\lambda \end{vmatrix} = (\lambda-9)(\lambda-1)+16$$
$$= \lambda^2 - 10\lambda + 9 + 16$$
$$= \lambda^2 - 10\lambda + 25$$
$$= (\lambda-5)^2$$

$\lambda = 5, 5$

$$(A-\lambda I)V = \begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad V = a\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$(A-\lambda I)^2 V_2 = 0$$

$$\begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\text{set } V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \therefore \begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix} = V_1$$

$$\therefore \vec{x} = c_1 \vec{v}_1 e^{5t} + c_2(t\vec{v}_1 + \vec{v}_2)e^{5t}$$

+12

$$= c_1\begin{bmatrix} -4 \\ 4 \end{bmatrix}e^{5t} + c_2\begin{bmatrix} -4t+1 \\ 4t \end{bmatrix}e^{5t}$$

b) $$x' = \begin{bmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{bmatrix}x \qquad \det(A-\lambda I) = \begin{vmatrix} -2-\lambda & -9 & 0 \\ 1 & 4-\lambda & 0 \\ 1 & 3 & 1-\lambda \end{vmatrix}$$

$$\det(A-\lambda I) = (-2-\lambda)(4-\lambda)(1-\lambda) + 9(1-\lambda)$$
$$= (\lambda^2 - 2\lambda - 8 + 9)(1-\lambda) = (1-\lambda)^3 = 0$$

$\lambda = 1, 1, 1$

$$\begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad v = b\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

There are two independent eigenvector $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix}\begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ set $V_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\therefore \begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = V_1$

$V_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ so that V_1 V_2 V_3 are independent

$\therefore \vec{x} = C_1 \vec{V}_1 e^t + C_2(\vec{V}_1 t + \vec{V}_2)e^t + C_3 \vec{V}_3 e^t$

$= C_1 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} e^t + C_2 \begin{bmatrix} -3t+1 \\ t \\ t \end{bmatrix} e^t + C_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^t$ +12

8 a) $A = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}$ $A^2 = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore$ A is nilpotent

$e^{At} = I + At + \frac{A^2t^2}{2!} \cdots$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 3t & -t \\ 9t & -3t \end{pmatrix} = \begin{pmatrix} 3t+1 & -t \\ 9t & -3t+1 \end{pmatrix}$ +5

b) $x' = Ax + f(t)$ $A = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}$ $f(t) = \begin{pmatrix} 0 \\ t^{-2} \end{pmatrix}$ $x(1) = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$

$e^{-At} f(t) = \begin{pmatrix} 1-3t & t \\ -9t & 1+3t \end{pmatrix} \begin{pmatrix} 0 \\ t^{-2} \end{pmatrix} = \begin{pmatrix} t^{-1} \\ \frac{1+3t}{t^2} \end{pmatrix}$

$\int_1^t e^{-As} f(s)\, ds = \int_1^t \begin{pmatrix} s^{-1} \\ \frac{1+3s}{s^2} \end{pmatrix} ds = \begin{bmatrix} \ln s \\ (-1)s^{-1} + 3\ln s \end{bmatrix}_1^t$

$= \begin{bmatrix} \ln t - 0 \\ -\frac{1}{t} + 1 + 3\ln t \end{bmatrix}$

$e^{At} \int_1^t e^{-As} f(s)\, ds = \begin{bmatrix} 3t+1 & -t \\ 9t & -3t+1 \end{bmatrix} \begin{bmatrix} \ln t \\ -\frac{1}{t} + 1 + 3\ln t \end{bmatrix}$

$= \begin{bmatrix} 3t\ln t + \ln t + 1 - t - 3t\ln t \\ 9t\ln t + 3 - 3t - 9t\ln t + 1 - \frac{1}{t} + 3\ln t \end{bmatrix}$

$= \begin{bmatrix} \ln t + 1 - t \\ 4 - 3t - \frac{1}{t} + 3\ln t \end{bmatrix}$

$$e^{A(t-1)} = \begin{bmatrix} 3(t-1)+1 & -(t-1) \\ 9(t-1) & -3(t-1)+1 \end{bmatrix} = \begin{pmatrix} 3t-2 & 1-t \\ 9t-9 & -3t+4 \end{pmatrix}$$

$$e^{A(t-1)}x(1) = \begin{pmatrix} 3t-2 & 1-t \\ 9t-9 & -3t+4 \end{pmatrix}\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 9t-6+7-7t \\ 27t-27-21t+28 \end{pmatrix} = \begin{pmatrix} 2t+1 \\ 6t+1 \end{pmatrix}$$

$$\therefore x(t) = \begin{bmatrix} 2t+1 \\ 6t+1 \end{bmatrix} + \begin{bmatrix} \ln t + 1 - t \\ 4 - 3t - \frac{1}{t} + 3\ln t \end{bmatrix}$$

$$= \begin{bmatrix} \ln t + t + 2 \\ 5 + 3t - \frac{1}{t} + 3\ln t \end{bmatrix}.$$

+10