SOME PROPERTIES OF ORESME NUMBERS AND CONVOLUTIONS WITH VARIOUS SECOND ORDER SEQUENCES

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Abstract. The Cauchy convolution for the Oresme numbers and Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, and Jacobsthal-Lucas numbers are investigated. The convolutions leading to various congruences with expressions involving Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, and Jacobsthal-Lucas numbers are studied.

1. Introduction

In 1974, Horadam [8] indicated that, in spite of their considerable biological interest, the Oresme numbers had not received much attention. This is no longer the case. The biographies of Nicole Oresme [10, 12] and several papers [1, 3, 5, 6, 14] have addressed Oresme numbers and their generalizations. Additional information can be found at the *On-Line Encyclopedia of Integer Sequences* (OEIS) [13, A273692].

Many papers have been written on convolutions of sequences. Hoggatt [7] considered Fibonacci convolutions and Koshy [11] considered Pell and Pell-Lucas convolutions. In this paper, convolutions with Oresme numbers and Fibonacci, Lucas, Pell, and Jacobsthal sequences are investigated.

Definition 1.1. *The Oresme numbers* [13, A273692]

$$\dots,\ -896,\ -384,\ -160,\ -64,\ -24,\ -8,\ -2,\ 0,\ \frac{1}{2},\ \frac{2}{4},\ \frac{3}{8},\ \frac{4}{16},\ \frac{5}{32},\ \frac{6}{64},\ \frac{7}{128},\ \dots$$

are given by the initial conditions $O_0 = 0$, $O_1 = O_2 = \frac{1}{2}$, and the second order relation

$$O_{n+2} = O_{n+1} - \frac{1}{4}O_n,$$

with the closed form

$$O_n = n2^{-n}.$$

The Cauchy convolution $\{c_n\}$ for two sequences $\{a_n\}$ and $\{b_n\}$ is defined in [7].

Definition 1.2. *Let $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ be given. The Cauchy convolution $\{c_n\}_{n\geq 0}$ is defined by*

$$c_n = a_0b_n + a_1b_{n-1} + \cdots + a_{n-1}b_1 + a_nb_0 = \sum_{k=0}^{n} a_kb_{n-k},$$

where $n \geq 0$.

2. Convolutions of Oresme Numbers

Next, we state and prove some propositions regarding convolutions of Oresme numbers.

Proposition 2.1. *Let n be a nonnegative integer. Then*

$$\sum_{k=0}^{n} O_k O_{n-k} = \frac{(n+1)n(n-1)}{6 \cdot 2^n}.$$

Proof.

$$\begin{aligned}\sum_{k=0}^{n} O_k O_{n-k} &= \sum_{k=0}^{n} \frac{k}{2^n} \cdot \frac{n-k}{2^{n-k}} = \sum_{k=0}^{n} \frac{k(n-k)}{2^n} \\ &= \frac{n}{2^n} \sum_{k=0}^{n} k - \frac{1}{2^n} \sum_{k=1}^{n} k^2 = \frac{n}{2^n} \cdot \frac{n(n+1)}{2} - \frac{1}{2^n} \cdot \frac{n(n+1)(2n+1)}{6} \\ &= \frac{n}{2^n} \cdot \frac{n+1}{2} \left(\frac{3n-(2n+1)}{3} \right) = \frac{n}{2^n} \cdot \frac{n+1}{2} \cdot \frac{n-1}{3} = \frac{(n+1)n(n-1)}{6 \cdot 2^n}.\end{aligned}$$

□

3. Preliminary Tools

The key to computing the convolutions of Oresme numbers with Fibonacci, Lucas, Pell, and Pell-Lucas numbers is the following equation

$$\sum_{k=0}^{n} a_k O_{n-k} = \sum_{k=0}^{n} \frac{n-k}{2^{n-k}} a_k = \frac{n}{2^n} \sum_{k=0}^{n} 2^k a_k - \frac{1}{2^n} \sum_{k=0}^{n} k 2^k a_k, \tag{3.1}$$

where n is a nonnegative integer. This reduces the problem of computing the convolutions with Oresme numbers to computing the two sums on the right side of (3.1). In addition, for n a nonnegative integer, recall that

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1} = \frac{1-x^{n+1}}{1-x}, \tag{3.2}$$

and

$$\sum_{k=0}^{n} k x^k = x \left(\frac{(n+1)x^n}{x-1} - \frac{x^{n+1}-1}{(x-1)^2} \right) = \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}. \tag{3.3}$$

These formulas can be found in *Concrete Mathematics* [4, pp. 32–33]. With (3.1), (3.2), and (3.3), we can proceed to compute convolutions with Oresme numbers.

4. Convolutions of Fibonacci and Lucas Numbers with Oresme Numbers

Proposition 4.1. *Let n be a nonnegative integer and F_n be the nth Fibonacci number. Then*

$$\sum_{k=0}^{n} F_k O_{n-k} = \frac{2}{5} F_{n+1} - \frac{2(n+1)}{5 \cdot 2^n}.$$

Proof. Let n be a nonnegative integer and let $0 \le k \le n$ be an integer. From the Binet formula for F_k, with $\alpha, \beta = \frac{1 \pm \sqrt{5}}{2}$, we have

$$F_k = \frac{\alpha^k - \beta^k}{\sqrt{5}}.$$

Thus,

$$\begin{aligned}\sum_{k=0}^{n} F_k O_{n-k} &= \sum_{k=0}^{n} \frac{n-k}{2^{n-k}} \cdot \frac{\alpha^k - \beta^k}{\sqrt{5}} \\ &= \frac{n}{2^n\sqrt{5}} \sum_{k=0}^{n} \left((2\alpha)^k - (2\beta)^k\right) - \frac{1}{2^n\sqrt{5}} \sum_{k=0}^{n} \left(k(2\alpha)^k - k(2\beta^k)\right).\end{aligned}$$

For readability, we compute the two sums separately and combine them later. For the first sum, we use (3.2) and that $2\alpha - 1 = \sqrt{5}$ and $2\beta - 1 = -\sqrt{5}$. Therefore,

$$\begin{aligned}\frac{n}{2^n\sqrt{5}} \sum_{k=0}^{n} \left((2\alpha)^k - (2\beta)^k\right) &= \frac{n}{2^n\sqrt{5}} \left(\frac{2^{n+1}\alpha^{n+1} - 1}{2\alpha - 1} - \frac{2^{n+1}\beta^{n+1} - 1}{2\beta - 1}\right) \\ &= \frac{2n}{\sqrt{5}} \left(\frac{\alpha^{n+1}}{\sqrt{5}} - \frac{\beta^{n+1}}{-\sqrt{5}}\right) + \frac{n}{2^n\sqrt{5}} \left(\frac{-1}{\sqrt{5}} - \frac{-1}{-\sqrt{5}}\right) \qquad (4.1) \\ &= \frac{2n}{5}(\alpha^{n+1} + \beta^{n+1}) + \frac{n}{2^n\sqrt{5}} \cdot \frac{-2}{\sqrt{5}} \\ &= \frac{2n}{5} L_{n+1} - \frac{2n}{5 \cdot 2^n} \\ &= \frac{2n}{5}(F_n + F_{n+2}) - \frac{2n}{5 \cdot 2^n}.\end{aligned}$$

For the second sum, we use (3.3) and that $(2\alpha - 1)^2 = (2\beta - 1)^2 = 5$. Therefore,

$$\begin{aligned}&\sum_{k=0}^{n} \left(k(2\alpha)^k - k(2\beta)^k\right) \\ &= 2\alpha \left(\frac{(n+1)2^n\alpha^n}{2\alpha - 1} - \frac{2^{n+1}\alpha^{n+1} - 1}{(2\alpha - 1)^2}\right) - 2\beta \left(\frac{(n+1)2^n\beta^n}{2\beta - 1} - \frac{2^{n+1}\beta^{n+1} - 1}{(2\beta - 1)^2}\right) \\ &= \frac{(n+1)2^{n+1}\alpha^{n+1}}{\sqrt{5}} - \frac{2^{n+2}\alpha^{n+2} - 2\alpha}{5} + \frac{(n+1)2^{n+1}\beta^{n+1}}{\sqrt{5}} + \frac{2^{n+2}\beta^{n+2} - 2\beta}{5} \\ &= \frac{(n+1)2^{n+1}}{\sqrt{5}}(\alpha^{n+1} + \beta^{n+1}) - \frac{2^{n+2}}{5}(\alpha^{n+2} - \beta^{n+2}) + \frac{2}{5}(\alpha - \beta) \\ &= \frac{(n+1)2^{n+1}}{\sqrt{5}} L_{n+1} - \frac{2^{n+2}}{\sqrt{5}} F_{n+2} + \frac{2}{\sqrt{5}} \\ &= \frac{n2^{n+1}}{\sqrt{5}}(F_n + F_{n+2}) + \frac{2^{n+1}}{\sqrt{5}}(F_n + F_{n+2}) - \frac{2^{n+2}}{\sqrt{5}} F_{n+2} + \frac{2}{\sqrt{5}}.\end{aligned}$$

Thus,

$$-\frac{1}{2^n\sqrt{5}} \sum_{k=0}^{n} \left(k(2\alpha)^k - k(2\beta)^k\right) = -\frac{2n}{5}(F_n + F_{n+2}) - \frac{2}{5}(F_n + F_{n+2}) + \frac{2}{5}(2F_{n+2}) - \frac{2}{5} \cdot \frac{1}{2^n}. \quad (4.2)$$

Combining (4.1) and (4.2) yields

$$\begin{aligned}\sum_{k=0}^{n} F_k O_{n-k} &= -\frac{2n}{5\cdot 2^n} - \frac{2}{5}(F_n + F_{n+2}) + \frac{2}{5}(2F_{n+2}) - \frac{2}{5}\cdot\frac{1}{2^n}\\ &= \frac{2}{5}\left(2F_{n+2} - F_n - F_{n+2} - \left(\frac{n+1}{2^n}\right)\right)\\ &= \frac{2}{5}\left(F_{n+1} - 2\left(\frac{n+1}{2^{n+1}}\right)\right) = \frac{2}{5}F_{n+1} - \frac{2(n+1)}{5\cdot 2^n}.\end{aligned}$$

□

For $n = 0, 1, 2, 3, 4, 5, 6, 7$, the first eight values of the $F_{n+1} - \frac{n+1}{2^n}$ are

$$0,\ 0,\ \frac{5}{4},\ \frac{5}{2},\ \frac{75}{16},\ \frac{125}{16},\ \frac{825}{64},\ \frac{335}{16}.$$

All the numerators of these fractions have 5 as a factor. This leads us to the following corollary.

Corollary 4.2. *Let n be a nonnegative integer. Then* $2^{n-1}F_n \equiv n \pmod 5$.

Proof. The proof is by induction on n. The base step is true for $n = 0, 1, 2, 3, 4, 5, 6, 7$. Now assume that $n \geq 7$ and the corollary is true for all nonnegative integers $0, 1, \ldots, n$. Then

$$\begin{aligned}2^n F_{n+1} - (n+1) &= 2^n(F_n + F_{n-1}) - (n+1)\\ &= 2^n F_n + (2n - 2n) + 2^n F_{n-1} + (-4(n-1) + 4(n-1)) - (n+1)\\ &= (2^n F_n - 2n) + (2^n F_{n-1} - 4(n-1)) + 2n + 4(n-1) - (n+1)\\ &= 2(2^{n-1}F_n - n) + 4(2^{n-2}F_{n-1} - (n-1)) + 2n + 4(n-1) - (n+1)\\ &= 2(2^{n-1}F_n - n) + 4(2^{n-2}F_{n-1} - (n-1)) + 5(n-1).\end{aligned}$$

Because 5 divides each term, 5 divides $2^n F_{n+1} - (n+1)$. Thus, the result is true for $n+1$. Therefore, by the principle of mathematical induction, the corollary is true. □

Proposition 4.3. *Let n be a nonnegative integer. Then*

$$\sum_{k=0}^{n} L_k O_{n-k} = \frac{2}{5}L_{n+1} - \frac{2}{5\cdot 2^n}.$$

The proof of this statement is similar to that of the Fibonacci convolution and is omitted.

Inspection of the first few terms reveals, similar to Corollary 4.2, that 5 is a factor of the numerators of $L_{n+1} - \frac{1}{2^n}$.

Corollary 4.4. *Let n be a nonnegative integer. Then* $2^n L_{n+1} \equiv 1 \pmod 5$.

Proof. The proof is by induction on n. Inspection of the numerators of $L_{n+1} - \frac{1}{2^n}$ for $n = 0, 1, 2, 3, 4$ are $0,\ \frac{5}{2},\ \frac{15}{4},\ \frac{55}{8},\ \frac{175}{16}$. So the corollary is true for the first five terms. Let $n \geq 4$ and assume the induction hypothesis is true for all integers less than or equal to n. Then

$$\begin{aligned}2^{n+1}L_{n+2} - 1 &= 2^{n+1}(L_{n+1} + L_n) - 1\\ &= 2(2^n L_{n+1} - 1) + 4(2^{n-1}L_n - 1) + 2 + 4 - 1\\ &= 2(2^n L_{n+1} - 1) + 4(2^{n-1}L_n - 1) + 5.\end{aligned}$$

Again, because 5 divides each term, $2^{n+1}L_{n+2} \equiv 1 \pmod 5$. Thus, the result is true for $n+1$. Therefore, by the principle of mathematical induction, the corollary is true. □

5. Convolutions of Pell and Pell-Lucas Numbers with Oresme Numbers

We next consider convolutions of Pell and Pell-Lucas numbers with Oresme numbers. We will state some properties that we need for our calculations.

In the case of the Pell-Lucas sequence [13, A001333], 1, 3, 7, 17, 41, 99, ..., there is a bit of a problem. The OEIS does not call this the Pell-Lucas sequence, but refers to it as the numerators of continued fraction convergents of $\sqrt{2}$. The sequence [13, A002203], 0, 2, 6, 14, 34, 82, 198, ..., is called the Pell companion numbers. Bicknell [2] defines the Pell-Lucas sequence as the even number sequence; however, Koshy [11] calls the Pell-Lucas sequence the odd number sequence. In this paper, we use the odd number sequence as the Pell-Lucas sequence.

Definition 5.1. *The Pell numbers are defined by $P_0 = 0$, $P_1 = 1$, and for $n \geq 2$*

$$P_n = 2P_{n-1} + P_{n-2}.$$

The Pell-Lucas numbers are defined by $Q_0 = 1$, $Q_1 = 1$, and for $n \geq 2$

$$Q_n = 2Q_{n-1} + Q_{n-2}.$$

The Binet formulas for the Pell and Pell-Lucas numbers are

$$P_n = \frac{\phi^n - \psi^n}{2\sqrt{2}} \text{ and } Q_n = \frac{\phi^n + \psi^n}{2},$$

where

$$\phi = 1 + \sqrt{2} \text{ and } \psi = 1 - \sqrt{2}.$$

Among other identities, note that

$$\phi \cdot \psi = -1, \quad \phi + \psi = 2, \text{ and } \phi - \psi = 2\sqrt{2}.$$

We now state the following two Propositions.

Proposition 5.2. *Let n be a nonnegative integer. Then*

$$\sum_{k=0}^{n} P_k O_{n-k} = \frac{2}{49}(5P_{n+1} - 4P_n) - \frac{2}{49}\left(\frac{7n+5}{2^n}\right).$$

Proof. Using (3.1) with a_k equal to P_k the first sum is found to be

$$\frac{n}{2^n}\sum_{k=0}^{n} 2^k P_k = \frac{14n}{49}\left(P_{n+1} + 2P_n - \frac{1}{2^n}\right),$$

and the second is

$$\frac{1}{2^n}\sum_{k=0}^{n} k2^k P_k = \frac{14n}{49}\left(P_{n+1} + 2P_n\right) - \frac{2}{49}(5P_{n+1} - 4P_n) + \frac{10}{49 \cdot 2^n}.$$

Subtracting the two expressions yields the statement of the theorem. □

Corollary 5.3. *Let n be a nonnegative integer, then*

$$2^n(5P_{n+1} - 4P_n) \equiv (7n + 5) \pmod{49}.$$

Proof. Again, the proof is by induction on n. The values of the congruence mod 49 on the left and right sides of the congruence for $n = 0$ and $n = 1$ are 5 and 12, respectively. Let $n \geq 1$ and assume that

$$2^n(5P_{n+1} - 4P_n) - (7n + 5) \equiv 0 \pmod{49} \text{ and } 2^{n-1}(5P_n - 4P_{n-1}) - (7n - 2) \equiv 0 \pmod{49}.$$

We will prove that the congruence is true for $n+1$. We have

$$\begin{aligned}
&2^{n+1}(5P_{n+2}-4P_{n+1})-(7n+12)\\
&=2^{n+1}(10P_{n+1}-3P_n-4P_{n-1})-(7n+12)\\
&=2^{n+2}(5P_{n+1}-4P_n)+2^{n+1}(5P_n-4P_{n-1})-4(7n+5)-4(7n-2)+49n\\
&=4\left(2^n(5P_{n+1}-4P_n)-(7n+5)\right)+4\left(2^{n-1}(5P_n-4P_{n-1})-(7n-2)\right)+49n.
\end{aligned}$$

Because 49 divides each term, 49 divides $2^{n+1}(5P_{n+2}-4P_{n+1})-(7n+12)$. Thus, the result is true for $n+1$. Therefore, by the principle of mathematical induction, the corollary is true. □

Proposition 5.4. *Let n be a nonnegative integer. Then*

$$\sum_{k=0}^{n} Q_k O_{n-k} = \frac{2}{49}(5Q_{n+1}-4Q_n)+\frac{1}{49}\left(\frac{7n-2}{2^n}\right).$$

The proof is analogous to that of the Pell case.

Corollary 5.5. *Let n be a nonnegative integer, then*

$$2^{n+1}(5Q_{n+1}-4Q_n)\equiv(2-7n)\equiv 0 \pmod{49}.$$

Proof. Again, the proof is by induction on n. The values of the congruence mod 49 on the left and right sides of the congruence for $n=0$ and $n=1$ are 2 and 44, respectively. Let $n\geq 1$ and assume that

$$2^{n+1}(5Q_{n+1}-4Q_n)-(2-7n)\equiv 0 \pmod{49} \text{ and } 2^n(5Q_n-4Q_{n-1})-(9-7n)\equiv 0 \pmod{49}.$$

We will prove that the congruence is true for $n+1$. We have

$$\begin{aligned}
&2^{n+2}(5Q_{n+2}-4Q_{n+1})-(-5-7n)\\
&=2^{n+2}(10Q_{n+1}-3Q_n-4Q_{n-1})-(-5-7n)\\
&=2^{n+3}(5Q_{n+1}-4Q_n)+2^{n+2}(5Q_n-4Q_{n-1})-4(2-7n)-4(9-7n)+49-49n\\
&=4\left(2^{n+1}(5Q_{n+1}-4Q_n)-(2-7n)\right)+4\left(2^n(5Q_n-4Q_{n-1})-(9-7n)\right)+49-49n.
\end{aligned}$$

Because 49 divides each term, 49 divides $2^{n+1}(5Q_{n+1}-4Q_n)-(-5-7n)$. Thus, the result is true for $n+1$. Therefore, by the principle of mathematical induction, the corollary is true. □

6. Convolutions of Jacobsthal and Jacobsthal-Lucas Numbers with Oresme Numbers

We next consider convolutions of Jacobsthal and Jacobsthal-Lucas numbers with Oresme numbers.

Definition 6.1. *The Jacobsthal numbers are defined by $J_0=0$, $J_1=1$, and for $n\geq 2$*

$$J_n = J_{n-1}+2J_{n-2}.$$

The Jacobsthal-Lucas numbers are defined by $j_0=2$, $j_1=1$, and for $n\geq 2$

$$j_n = j_{n-1}+2j_{n-2}.$$

Proposition 6.2. *Let n be a nonnegative integers. Then*

$$\sum_{k=0}^{n} J_k O_{n-k} = \frac{2}{9} J_{n+1} - \frac{2(n+1)}{9 \cdot 2^n}.$$

Proof. Note that the closed form of J_k is $\frac{2^k - (-1)^k}{3}$ so

$$\sum_{k=0}^{n} J_k O_{n-k} = \frac{1}{3} \sum_{k=0}^{n} 2^k O_{n-k} - \frac{1}{3} \sum_{k=0}^{n} (-1)^k O_{n-k}.$$

By (3.1) we have

$$\sum_{k=0}^{n} 2^k O_{n-k} = \frac{n}{2^n} \sum_{k=0}^{n} 2^{2k} - \frac{1}{2^n} \sum_{k=0}^{n} k 2^{2k},$$

which by (3.2) and (3.3) is

$$\frac{n}{2^n} \left(\frac{2^{2n+2} - 1}{n-1} \right) - \frac{1}{2^n} \cdot 4 \cdot \left(\frac{(n+1)2^{2n+2}}{3} - \frac{2^{2n+2} - 1}{9} \right).$$

By [8] we have

$$\begin{aligned} \sum_{k=0}^{n} (-1)^k O_{n-k} = (-1)^n \sum_{k=0}^{n} (-1)^{-k} O_{n-k} &= (-1)^n \sum_{k=0}^{n} (-1)^k O_k \\ &= (-1)^n \cdot \frac{4}{9} \left(\frac{-1}{2} + (-1)^n O_{n+2} - 2 O_{n+1} \right). \end{aligned}$$

With tedious expansion, collecting like terms, using the closed form of the Jacobsthal numbers, and dividing by a further third, yields the result. □

We next have a corollary similar to Corollary 4.2. For $n = 0, 1, 2, 3, 4$, the first five values of the $J_{n+1} - 2O_{n+1}$ are

$$0, \ 0, \ \frac{9}{4}, \ \frac{9}{2}, \ \frac{171}{16}.$$

All of the numerators of these fractions have 9 as a factor. This leads us to the following corollary.

Corollary 6.3. *Let n be a nonnegative integer. Then $2^n J_{n+1} \equiv (n+1) \pmod 9$.*

Proof. The proof is by induction on n. The base step is true for $n = 0, 1, 2, 3, 4$. Now, assume that $n \geq 4$ and the corollary is true for all nonnegative integers $0, 1, \ldots, n$. Then

$$\begin{aligned} 2^{n+1} J_{n+2} - (n+2) &= 2^{n+1}(J_{n+1} + 2J_n) - (n+2) \\ &= 2(2^n J_{n+1} - (n+1)) + 8(2^{n-1} J_n - n) + 9n. \end{aligned}$$

Because 9 divides each term, $2^{n+1} J_{n+2} \equiv (n+2) \pmod 9$. Thus, the result is true for $n+1$. Therefore, by the principle of mathematical induction, the corollary is true. □

Proposition 6.4. *Let n be a nonnegative integers. Then*

$$\sum_{k=0}^{n} j_k O_{n-k} + \frac{2}{9} j_{n+1} - \frac{2}{9 \cdot 2^n}.$$

Corollary 6.5. *Let n be a nonnegative integer. Then $2^n j_{n+1} \equiv 1 \pmod 9$.*

7. Concluding Comments

It is interesting to note that formula (3.1), which was used to find convolutions for second order sequences with the Oresme sequence, is not limited to second order sequences. Readers might find it interesting to consider a few third order sequences. Many such sequences exist; the most recognized ones being the Perrin [13, A0001608], Padovan [13, A000931], Tribonacci [13, A000073], and the Narayana Cow [13, A000930] sequences.

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