NOTE ON THE GENERALIZED LEONARDO NUMBERS

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Abstract. The dual generalized Leonardo sequence, a sequence related to the generalized Leonardo sequence, is introduced. These sequences contain many previously known extensions of the Fibonacci and Lucas sequences. Properties of the generalized Leonardo and dual generalized Leonardo sequences are derived.

1. Introduction

The sequences of Fibonacci $\{F_n\}_{n\geq 0}$ and Lucas $\{L_n\}_{n\geq 0}$ numbers are defined by

$$F_0 = 0,\ F_1 = 1, \quad \text{and} \quad F_{n+1} = F_n + F_{n-1} \quad (n \geq 1);$$

$$L_0 = 2,\ L_1 = 1, \quad \text{and} \quad L_{n+1} = L_n + L_{n-1} \quad (n \geq 1).$$

The Fibonacci and Lucas numbers are related by the identities

$$F_{n-1} + F_{n+1} = L_n \quad \text{and} \quad L_{n-1} + L_{n+1} = 5F_n.$$

Catarino and Borges [2] defined the sequence of Leonardo numbers $\{Le_n\}_{n\geq 0}$ by

$$Le_0 = Le_1 = 1 \quad \text{and} \quad Le_{n+1} = Le_n + Le_{n-1} + 1 \quad (n \geq 1)$$

and gave some properties of this sequence in [1, 2, 3, 6]. The author and Chobsorn [5] presented the generalized Leonardo sequence $\{\mathcal{L}_{k,n}\}_{n\geq 0}$,

$$\mathcal{L}_{k,0} = \mathcal{L}_{k,1} = 1 \quad \text{and} \quad \mathcal{L}_{k,n+1} = \mathcal{L}_{k,n} + \mathcal{L}_{k,n-1} + k \quad (n \geq 1),$$

where k is a fixed positive integer. When $k = 1$, this sequence is the Leonardo sequence.

The nonhomogeneous recurrence relation of the generalized Leonardo numbers can be rewritten as the homogenous recurrence relation

$$\mathcal{L}_{k,n+1} = 2\mathcal{L}_{k,n} - \mathcal{L}_{k,n-2}.$$

The relationship between Fibonacci and generalized Leonardo numbers is expressed in the identity [5]

$$\mathcal{L}_{k,n} = (k+1)F_{n+1} - k. \tag{1.1}$$

When m, n, and t are nonnegative integers, the following identities for Fibonacci numbers are known [4, Chapter 5].

$$F_mF_n - F_{m+t}F_{n-t} = (-1)^{n-t}F_{m-n+t}F_t, \tag{1.2}$$

$$F^2_{m+n} - F^2_{m-n} = F_{2m}F_{2n}, \tag{1.3}$$

$$F^2_{m+n+1} + F^2_{m-n} = F_{2m+1}F_{2n+1}. \tag{1.4}$$

In this work, we begin by introducing the dual generalized Leonardo sequence and derive relationships of the generalized Leonardo and dual generalized Leonardo numbers, and give similar identities (1.2)–(1.4) for the generalized Leonardo numbers. Then, we show the sums

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of products of the binomial coefficients and the generalized Leonardo or dual generalized Leonardo numbers.

2. Definition and Basic Properties

We start with a definition of a new sequence and establish its basic properties.

Definition 2.1. *For a fixed nonnegative integer k, the dual generalized Leonardo sequence $\{\mathcal{M}_{k,n}\}_{n\geq 0}$ is defined as*

$$\mathcal{M}_{k,n+1} = \mathcal{M}_{k,n} + \mathcal{M}_{k,n-1} + 2k \quad (n \geq 1),$$

with initial conditions $\mathcal{M}_{k,0} = 1 - k$ and $\mathcal{M}_{k,1} = k + 3$.

For $k = 1$, we call $\mathcal{M}_{1,n}$ the dual Leonardo numbers, denoted by M_n.

The first few terms of the generalized Leonardo sequence and the dual generalized Leonardo sequence for $k = 1, 2, 3, 4$ are shown in the following table.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Le_n	1	1	3	5	9	15	25	41	67	109	177	287	465	753	1219
$\mathcal{L}_{2,n}$	1	1	4	7	13	22	37	61	100	163	265	430	697	1129	1828
$\mathcal{L}_{3,n}$	1	1	5	9	17	29	49	81	133	217	353	573	929	1505	2437
$\mathcal{L}_{4,n}$	1	1	6	11	21	36	61	101	166	271	441	716	1161	1881	3046
M_n	0	4	6	12	20	34	56	92	150	244	396	642	1040	1684	2726
$\mathcal{M}_{2,n}$	−1	5	8	17	29	50	83	137	224	365	593	962	1559	2525	4088
$\mathcal{M}_{3,n}$	−2	6	10	22	38	66	110	182	298	486	790	1282	2078	3366	5450
$\mathcal{M}_{4,n}$	−3	7	12	27	47	82	137	227	372	607	987	1602	2597	4207	6812

We can prove that

$$\mathcal{L}_{k,n-1} + \mathcal{L}_{k,n+1} = \mathcal{M}_{k,n}, \tag{2.1}$$

and

$$\mathcal{M}_{k,n-1} + \mathcal{M}_{k,n+1} = 5\mathcal{L}_{k,n} + k. \tag{2.2}$$

By (2.1) and the definition of the generalized Leonardo numbers, we get

$$\mathcal{M}_{k,n} - 2\mathcal{L}_{k,n-1} = \mathcal{L}_{k,n} + k. \tag{2.3}$$

By (1.1) and (2.1), we obtain a connection between the dual generalized Leonardo and the Lucas numbers, i.e.,

$$\mathcal{M}_{k,n} = (k+1)L_{n+1} - 2k. \tag{2.4}$$

We will give some connections between the generalized Leonardo numbers and the dual generalized Leonardo numbers that are extensions of the Fibonacci and Lucas sequences. The following identities are known.

$$F_nL_n = F_{2n}, \tag{2.5}$$

$$F_{n+1}L_{n+2} - F_{n+2}L_n = F_{2n+1}, \tag{2.6}$$

$$L_n^2 - F_n^2 = 4F_{n-1}F_{n+1}. \tag{2.7}$$

Theorem 2.2. *Let n be a nonnegative integer. Then*

$$\mathcal{L}_{k,n}\mathcal{M}_{k,n} = (k+1)\mathcal{L}_{k,2n+1} - k\mathcal{L}_{k,n+3} + k(2k+1). \tag{2.8}$$

Proof. Using (1.1), (2.4), and (2.5), we get

$$\begin{aligned}\mathcal{L}_{k,n}\mathcal{M}_{k,n} &= ((k+1)F_{n+1}-k)((k+1)L_{n+1}-2k)\\ &= (k+1)^2F_{n+1}L_{n+1}-k(k+1)(2F_{n+1}+L_{n+1})+2k^2\\ &= (k+1)^2F_{2n+2}-k(k+1)F_{n+4}+2k^2\\ &= (k+1)\mathcal{L}_{k,2n+1}-k\mathcal{L}_{k,n+3}+k(2k+1),\end{aligned}$$

as desired. □

Theorem 2.3. *Let n be a nonnegative integer. Then*

$$\mathcal{L}_{k,n}\mathcal{M}_{k,n+1}-\mathcal{L}_{k,n+1}\mathcal{M}_{k,n-1}=(k+1)\mathcal{L}_{k,2n}-k\mathcal{L}_{k,n}+k. \tag{2.9}$$

Proof. Using (1.1), (2.4), and (2.6), we obtain

$$\begin{aligned}\mathcal{L}_{k,n}\mathcal{M}_{k,n+1}-\mathcal{L}_{k,n+1}\mathcal{M}_{k,n-1} &= (k+1)^2F_{2n+1}-k(k+1)L_{n+1}+2k(k+1)F_n\\ &= (k+1)\mathcal{L}_{k,2n}+2k\mathcal{L}_{k,n-1}-k\mathcal{M}_{k,n}+k(k+1).\end{aligned}$$

Using (2.3), we get the desired result. □

Theorem 2.4. *Let n be a nonnegative integer. Then*

$$\mathcal{M}_{k,n}^2-\mathcal{L}_{k,n}^2=4\mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1}+2k\mathcal{L}_{k,n}+k^2. \tag{2.10}$$

Proof. Observing that

$$(k+1)^2F_nF_{n+2}=(\mathcal{L}_{k,n-1}+k)(\mathcal{L}_{k,n+1}+k)=\mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1}+k\mathcal{M}_{k,n}+k^2,$$

we get, using (1.1), (2.4), (2.7), and above identity, that

$$\begin{aligned}\mathcal{M}_{k,n}^2-\mathcal{L}_{k,n}^2 &= (k+1)^2(L_{n+1}^2-F_{n+1}^2)+2k(k+1)(F_{n+1}-2L_{n+1})+3k^2\\ &= 4(k+1)^2F_nF_{n+2}+2k(\mathcal{L}_{k,n}-2\mathcal{M}_{k,n}-3k)+3k^2\\ &= 4\mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1}+2k\mathcal{L}_{k,n}+k^2,\end{aligned}$$

which is what we wanted to prove. □

For $k=1$, we get the following identities.

(1) $Le_{n-1}+Le_{n+1}=M_n$.
(2) $M_{n-1}+M_{n+1}=5Le_n+1$.
(3) $Le_nM_n=2Le_{2n+1}-Le_{n+3}+3$.
(4) $Le_nM_{n+1}-Le_{n+1}M_{n-1}=2Le_{2n}-Le_n+1$.
(5) $M_n^2-Le_n^2=4Le_{n-1}Le_{n+1}+2Le_n+1$.

The following theorem is a generalized Catalan's identity for the generalized Leonardo numbers.

Theorem 2.5. *Let $m,n\geq 1$ and $t\geq 0$ be integers. Then*

$$\begin{aligned}&\mathcal{L}_{k,m}\mathcal{L}_{k,n}-\mathcal{L}_{k,m+t}\mathcal{L}_{k,n-t}\\ &=(-1)^{n-t+1}(\mathcal{L}_{k,m-n+t-1}+k)(\mathcal{L}_{k,t-1}+k)-k(\mathcal{L}_{k,m}+\mathcal{L}_{k,n}-\mathcal{L}_{k,m+t}-\mathcal{L}_{k,n-t}).\end{aligned} \tag{2.11}$$

Proof. Consider

$$\begin{aligned}\mathcal{L}_{k,m}\mathcal{L}_{k,n} &= ((k+1)F_{m+1}-k)\,((k+1)F_{n+1}-k)\\ &= (k+1)^2F_{m+1}F_{n+1}-k(k+1)(F_{m+1}+F_{n+1})+k^2\\ &= (k+1)^2F_{m+1}F_{n+1}-k(\mathcal{L}_{k,m}+\mathcal{L}_{k,n})-k^2.\end{aligned}$$

Using (1.2), we obtain that

$$\mathcal{L}_{k,m}\mathcal{L}_{k,n} - \mathcal{L}_{k,m+t}\mathcal{L}_{k,n-t} = (k+1)^2(-1)^{n-t+1}F_{m-n+t}F_t - k\left(\mathcal{L}_{k,m} + \mathcal{L}_{k,n} - \mathcal{L}_{k,m+t} - \mathcal{L}_{k,n-t}\right).$$

Hence, by (1.1), Theorem 2.5 is proved. □

Taking $m = n$, identity (2.11) becomes

$$\mathcal{L}_{k,n}^2 - \mathcal{L}_{k,n+t}\mathcal{L}_{k,n-t} = (-1)^{n-t+1}(\mathcal{L}_{k,t-1} + k)^2 - k(2\mathcal{L}_{k,n} - \mathcal{L}_{k,n+t} - \mathcal{L}_{k,n-t}). \quad (2.12)$$

Putting $t = 1$ in (2.12), we obtain Cassini's identity for the generalized Leonardo numbers

$$\mathcal{L}_{k,n}^2 - \mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1} = \begin{cases} k\mathcal{L}_{k,n-3} - (2k+1), & \text{if } n \text{ odd;} \\ k\mathcal{L}_{k,n-3} + 2k(k+1) + 1, & \text{if } n \text{ even.} \end{cases} \quad (2.13)$$

Taking $k = 1$ in (2.13), we obtain Cassini's identity for Leonardo numbers (see [1])

$$Le_n^2 - Le_{n-1}Le_{n+1} = \begin{cases} Le_{n-3} - 3, & \text{if } n \text{ odd;} \\ Le_{n-3} + 5, & \text{if } n \text{ even.} \end{cases}$$

Next, the following theorem gives identities similar to (1.3) and (1.4) for the generalized Leonardo numbers.

Theorem 2.6. *Let m and n be two positive integers. Then*

(i) $\mathcal{L}_{k,m+n}^2 - \mathcal{L}_{k,m-n}^2 = (\mathcal{L}_{k,2m+1} + k)(\mathcal{L}_{k,2n-1} + k) - 2k(\mathcal{L}_{k,m+n} - \mathcal{L}_{k,m-n})$.

(ii) $\mathcal{L}_{k,m+n}^2 + \mathcal{L}_{k,m-n-1}^2 = (\mathcal{L}_{k,2m} + k)(\mathcal{L}_{k,2n} + k) - 2k(\mathcal{L}_{k,m+n} + \mathcal{L}_{k,m-n}) - 2k^2$.

Proof. Using identities (1.1) and (1.3), we have that

$$\begin{aligned} \mathcal{L}_{k,m+n}^2 - \mathcal{L}_{k,m-n}^2 &= ((k+1)F_{m+n+1} - k)^2 - ((k+1)F_{m-n+1} - k)^2 \\ &= (k+1)^2(F_{m+n+1}^2 - F_{m-n+1}^2) - 2k(k+1)(F_{m+n+1} - F_{m-n+1}) \\ &= (k+1)^2F_{2m+2}F_{2n} - 2k(\mathcal{L}_{k,m+n} - \mathcal{L}_{k,m-n}) \\ &= (\mathcal{L}_{k,2m+1} + k)(\mathcal{L}_{k,2n-1} + k) - 2k(\mathcal{L}_{k,m+n} - \mathcal{L}_{k,m-n}). \end{aligned}$$

The other assertion is proved similarly using (1.4). □

Taking $n = 1$ in Theorem 2.6, we obtain

$$\begin{aligned} \mathcal{L}_{k,m+1}^2 - \mathcal{L}_{k,m-1}^2 &= (k+1)\mathcal{L}_{k,2m+1} - 2k\mathcal{L}_{k,m} - k(k-1), \\ \mathcal{L}_{k,m+1}^2 + \mathcal{L}_{k,m}^2 &= (k+1)\mathcal{L}_{k,2m+2} - 2k\mathcal{L}_{k,m+2} + k(k+1), \end{aligned}$$

and for $k = 1$, we obtain

$$\begin{aligned} Le_{m+1}^2 - Le_{m-1}^2 &= 2Le_{2m+1} - 2Le_m, \\ Le_{m+1}^2 + Le_m^2 &= 2Le_{2m+2} - 2Le_{m+2} + 2. \end{aligned}$$

3. Binomial-Leonardo and Binomial-dual Leonardo Identities

We begin with some identities for the sums of products of binomial coefficients and the generalized Leonardo or dual generalized Leonardo numbers.

Theorem 3.1. *Let n and r be nonnegative integers. Then*

(i) $\displaystyle\sum_{i=0}^{n} \binom{n}{i} \mathcal{L}_{k,i+r} = \mathcal{L}_{k,2n+r} - k(2^n - 1)$.

(ii) $\sum_{i=0}^{n}\binom{n}{i}\mathcal{M}_{k,i+r}=\mathcal{M}_{k,2n+r}-2k(2^n-1)$.

(iii) $\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\mathcal{L}_{k,i+r}=(-1)^n\left(\mathcal{L}_{k,n-r-2}+k\right)$.

(iv) $\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\mathcal{M}_{k,i+r}=(-1)^{n-1}\left(\mathcal{M}_{k,n-r-2}+2k\right)$.

Proof. We use the following formulas of binomial Fibonacci and Lucas identities [4, page 162].

$$\sum_{i=0}^{n}\binom{n}{i}F_{i+r}=F_{2n+r}\quad\text{and}\quad\sum_{i=0}^{n}\binom{n}{i}L_{i+r}=L_{2n+r}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}F_{i+r}=(-1)^nF_{n-r}\quad\text{and}\quad\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}L_{i+r}=(-1)^{n-1}L_{n-r}.$$

Because the proofs of all the parts are similar, we only give the proofs for parts (i) and (iv). We get

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}\mathcal{L}_{k,i+r}&=\sum_{i=0}^{n}\binom{n}{i}((k+1)F_{i+r+1}-k)\\&=(k+1)F_{2n+r+1}-2^nk\\&=\mathcal{L}_{k,2n+r}-k(2^n-1).\end{aligned}$$

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\mathcal{M}_{k,i+r}&=\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}((k+1)L_{i+r+1}-2k)\\&=(-1)^{n-1}(k+1)L_{n-r-1}\\&=(-1)^{n-1}(\mathcal{M}_{k,n-r-2}+2k).\end{aligned}$$

This completes the proof. □

Theorem 3.2. *Let n and r be nonnegative integers. Then*

(i) $\sum_{i=0}^{n}\binom{n}{i}\mathcal{L}_{k,2i+r}=\begin{cases}5^{n/2}\mathcal{L}_{k,n+r}+k(5^{n/2}-2^n), & \text{if } n \text{ even;}\\ 5^{(n-1)/2}\mathcal{M}_{k,n+r+1}+2k(5^{(n-1)/2}-2^{n-1}), & \text{if } n \text{ odd.}\end{cases}$

(ii) $\sum_{i=0}^{n}\binom{n}{i}\mathcal{M}_{k,2i+r}=\begin{cases}5^{n/2}\mathcal{M}_{k,n+r}+2k(5^{n/2}-2^n), & \text{if } n \text{ even;}\\ 5^{(n+1)/2}\mathcal{L}_{k,n+r+1}+k(5^{(n+1)/2}-2^{n+1}), & \text{if } n \text{ odd.}\end{cases}$

Proof. In [4, page 163], the following formulas of binomial Fibonacci and Lucas identities are given.

$$\sum_{i=0}^{2n}\binom{2n}{i}F_{2i+r}=5^nF_{2n+r}\quad\text{and}\quad\sum_{i=0}^{2n+1}\binom{2n+1}{i}F_{2i+r}=5^nL_{2n+r+1},$$

$$\sum_{i=0}^{2n}\binom{2n}{i}L_{2i+r}=5^nL_{2n+r}\quad\text{and}\quad\sum_{i=0}^{2n+1}\binom{2n+1}{i}L_{2i+r}=5^{n+1}F_{2n+r+1}.$$

Because the proofs of parts (i) and (ii) are similar, we only give a proof for part (i). We get that

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}\mathcal{L}_{k,2i+r} &= \sum_{i=0}^{n}\binom{n}{i}((k+1)F_{i+r+1}-k)\\ &= \begin{cases}(k+1)5^{n/2}F_{n+r+1}-2^nk, & \text{if } n \text{ even}\\ (k+1)5^{(n-1)/2}L_{n+r+2}-2^nk, & \text{if } n \text{ odd}\end{cases}\\ &= \begin{cases}5^{n/2}\mathcal{L}_{k,n+r}+k(5^{n/2}-2^n), & \text{if } n \text{ even}\\ 5^{(n-1)/2}\mathcal{M}_{k,n+r+1}+2k(5^{(n-1)/2}-2^{n-1}), & \text{if } n \text{ odd.}\end{cases}\end{aligned}$$

This completes the proof of the theorem. □

We give six more similar sums.

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}F_{2i+r} = F_{n+r} \quad\text{and}\quad \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}L_{2i+r} = L_{n+r},$$

$$\sum_{i=0}^{n}\binom{n}{i}F_{3i+r} = 2^nF_{2n+r} \quad\text{and}\quad \sum_{i=0}^{n}\binom{n}{i}L_{3i+r} = 2^nL_{2n+r},$$

$$\sum_{i=0}^{n}\binom{n}{i}F_{4i+r} = 3^nF_{2n+r} \quad\text{and}\quad \sum_{i=0}^{n}\binom{n}{i}L_{4i+r} = 3^nL_{2n+r}.$$

We can use Binet's formulas for the Fibonacci and Lucas numbers to prove these identities. We will show the proof of only one identity.

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}F_{3i+r} &= \sum_{i=0}^{n}\binom{n}{i}\frac{\alpha^{3i+r}-\beta^{3i+r}}{\alpha-\beta}\\ &= \frac{1}{\alpha-\beta}\left(\alpha^r\sum_{i=0}^{n}\binom{n}{i}\alpha^{3i}-\beta^r\sum_{i=0}^{n}\binom{n}{i}\beta^{3i}\right)\\ &= \frac{1}{\alpha-\beta}\left(\alpha^r(1+\alpha^3)^n-\beta^r(1+\beta^3)^n\right)\\ &= \frac{1}{\alpha-\beta}\left(\alpha^r(2\alpha+2)^n-\beta^r(2\beta+2)^n\right)\\ &= \frac{2^n}{\alpha-\beta}\left(\alpha^{2n+r}-\beta^{2n+r}\right)\\ &= 2^nF_{2n+r},\end{aligned}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of equation $x^2-x-1=0$.

In a similar way, using the above identities, we get the following theorems.

Theorem 3.3. *Let n and r be nonnegative integers. Then*

(i) $\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\mathcal{L}_{k,2i+r} = \mathcal{L}_{k,n+r}+k.$

(ii) $\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\mathcal{M}_{k,2i+r} = \mathcal{M}_{k,n+r}+2k.$

Theorem 3.4. *Let n and r be nonnegative integers. Then*

(i) $\sum_{i=0}^{n} \binom{n}{i} \mathcal{L}_{k,3i+r} = 2^n \mathcal{L}_{k,2n+r}$.

(ii) $\sum_{i=0}^{n} \binom{n}{i} \mathcal{M}_{k,3i+r} = 2^n \mathcal{M}_{k,2n+r}$.

(iii) $\sum_{i=0}^{n} \binom{n}{i} \mathcal{L}_{k,4i+r} = 3^n \mathcal{L}_{k,2n+r} + k(3^n - 2^n)$.

(iv) $\sum_{i=0}^{n} \binom{n}{i} \mathcal{M}_{k,4i+r} = 3^n \mathcal{M}_{k,2n+r} + k(3^n - 2^n)$.

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