ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to Robert Frontczak, LBBW, Am Hauptbahnhof 2, 70173 Stuttgart, Germany or by email at the address robert.frontczak@lbbw.de. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-941 Proposed by Kunle Adegoke, Ile-Ife, Nigeria

Prove that

$$44\sum_{j=1}^{n} F_jF_{j+1}F_{j+2}F_{j+3}F_{j+4} = F_{n+2}F_{n+3}F_{n+5}\left(F_{n+4}\left(3F_n+2F_{n+2}\right)-5F_nF_{n+1}\right)-30\left(F_{n+4}-1\right)$$

and

$$44\sum_{j=1}^{n} L_jL_{j+1}L_{j+2}L_{j+3}L_{j+4} = L_{n+2}L_{n+3}L_{n+5}\left(L_{n+4}\left(3L_n+2L_{n+2}\right)-5L_nL_{n+1}\right)-750L_{n+4}-4518.$$

H-942 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, and Neculai Stanciu, Buzău, Romania

Show that

$$\liminf_{n\to\infty}\left(\int_{\sqrt[n]{n!L_n}}^{\sqrt[n+1]{(n+1)!L_{n+1}}}\sqrt[3]{\frac{\sin^6 x+\cos^6 x}{16}}dx\right)\geq\frac{\alpha}{4e},$$

with $\alpha=(1+\sqrt{5})/2$ being the golden ratio.

H-943 Proposed by Albert Stadler, Herrliberg, Switzerland

Put

$$\zeta_L(s)=\sum_{n=1}^{\infty}\frac{1}{\log^s(L_{2n})},\qquad Re(s)>1.$$

Prove the following:

(i) $\zeta_L(s)$ has an analytic continuation to the whole complex plane; the only singularity being a simple pole at $s=1$ with residue $\frac{1}{2\log\alpha}$.

(ii)

$$\zeta_L(0) = -\frac{1}{2},$$

$$\zeta_L(-1) = -\frac{1}{6}\log\alpha + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\frac{1}{\alpha^{4k}-1},$$

$$\zeta_L(-2) = 4\log\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\frac{\alpha^{4k}}{(\alpha^{4k}-1)^2} + 2\sum_{k=2}^{\infty}\frac{(-1)^k}{k}H_{k-1}\frac{1}{\alpha^{4k}-1},$$

where $H_k = \sum_{j=1}^{k} 1/j$ is the kth harmonic number.

H-944 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Prove that

(i) $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{F_{n+1}F_{n-1}F_{2n+1}F_{2n-1}} = \frac{5\sqrt{5}-11}{4}$;

(ii) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{L_{n+1}L_{n-1}F_{2n+1}F_{2n-1}} = \frac{2-\sqrt{5}}{4}$.

H-945 Proposed by the editor

Show that the Diophantine equation

$$F_n 2^{n-1} + 1 = m^2$$

has exactly four solutions in nonnegative integers n and m, namely, $(n,m) = (0,1)$, $(n,m) = (3,3)$, $(n,m) = (4,5)$, and $(n,m) = (5,9)$.

SOLUTIONS

H-911 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Let $r \geq 2$ and s be integers and $\mathbf{i} = \sqrt{-1}$. Prove that

(i) $\displaystyle\prod_{n=1}^{\infty}\left(1 + \frac{F_r}{F_{rn+s}}\right) = \frac{1+\beta^s}{1-\beta^{r+s}}$, if $s \geq 0$ is even;

(ii) $\displaystyle\prod_{n=1}^{\infty}\left(1 + \frac{F_r}{F_{rn+s}}\mathbf{i}\right) = \frac{\alpha^s+\mathbf{i}}{\alpha^s-\beta^r\mathbf{i}}$, if s is odd.

Solution by Brian Bradie, Newport News, VA

(i) Let $r \geq 2$ and $s \geq 0$ be even integers, and write

$$\begin{aligned} 1 + \frac{F_r}{F_{rn+s}} &= \frac{\alpha^{rn+s}-\beta^{rn+s}+\alpha^r-\beta^r}{\alpha^{rn+s}-\beta^{rn+s}}\cdot\frac{\beta^{rn+s}}{\beta^{rn+s}} \\ &= \frac{1-\beta^{2(rn+s)}+\beta^{r(n-1)+s}-\beta^{r(n+1)+s}}{1-\beta^{2(rn+s)}} \\ &= \frac{(1+\beta^{r(n-1)+s})(1-\beta^{r(n+1)+s})}{(1+\beta^{rn+s})(1-\beta^{rn+s})}. \end{aligned}$$

Thus, the product telescopes and is equal to

$$\lim_{n\to\infty}\frac{1+\beta^s}{1-\beta^{r+s}}\cdot\frac{1-\beta^{r(n+1)+s}}{1+\beta^{rn+s}} = \frac{1+\beta^s}{1-\beta^{r+s}},$$

because
$$\lim_{n\to\infty}\beta^{r(n+1)+s}=\lim_{n\to\infty}\beta^{rn+s}=0.$$

(ii) Let $r\geq 2$ be an even integer and s be an odd integer, and write
$$\begin{aligned}1+\frac{F_r}{F_{rn+s}}i &= \frac{\alpha^{rn+s}-\beta^{rn+s}+\alpha^r i-\beta^r i}{\alpha^{rn+s}-\beta^{rn+s}}\cdot\frac{\beta^{rn+s}}{\beta^{rn+s}}\\ &= \frac{-1-\beta^{2(rn+s)}+\beta^{r(n-1)+s}i-\beta^{r(n+1)+s}i}{-1-\beta^{2(rn+s)}}\\ &= \frac{1\ +\beta^{2(rn+s)}-\beta^{r(n-1)+s}i\ +\beta^{r(n+1)+s}i}{1\ +\beta^{2(rn+s)}}\\ &= \frac{(1-\beta^{r(n-1)+s}i)(1\ +\beta^{r(n+1)+s}i)}{(1-\beta^{rn+s}i)(1\ +\beta^{rn+s}i)}.\end{aligned}$$

Thus, the product telescopes and is equal to
$$\lim_{n\to\infty}\frac{1-\beta^s i}{1\ +\beta^{r+s}i}\cdot\frac{1\ +\beta^{r(n+1)+s}i}{1-\beta^{rn+s}i}=\frac{1-\beta^s i}{1\ +\beta^{r+s}i},$$

because
$$\lim_{n\to\infty}\beta^{r(n+1)+s}=\lim_{n\to\infty}\beta^{rn+s}=0.$$

Finally,
$$\prod_{n=1}^{\infty}\left(1+\frac{F_r}{F_{rn+s}}i\right)=\frac{1-\beta^s i}{1\ +\ \beta^{r+s}i}\cdot\frac{\alpha^s}{\alpha^s}=\frac{\alpha^s+i}{\alpha^s-\beta^r i}.$$

Also solved by Michel Bataille, Dmitry Fleischman, Ángel Plaza, Albert Stadler, Yunyong Zhang, and the proposer.

H-912 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Prove that

(i) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+4}}\left(\frac{1}{F_{n+1}}+\frac{1}{F_{n+2}}-\frac{1}{F_{n+3}}\right)=\frac{1}{3};$

(ii) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{F_nF_{n+4}}\left(\frac{1}{F_{n+1}}+\frac{1}{F_{n+2}}-\frac{1}{F_{n+3}}\right)=-\frac{1}{6};$

(iii) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}}\left(\frac{1}{F_{n+1}}+\frac{1}{F_{n+2}}-\frac{1}{F_{n+3}}\right)=\frac{1}{24}.$

Solution by Michel Bataille, Rouen, France

Let $K_n=\frac{1}{F_{n+1}}+\frac{1}{F_{n+2}}-\frac{1}{F_{n+3}}$. Using
$$F_{n+3}^2+F_nF_{n+3}=F_{n+3}\cdot 2F_{n+2}=F_{n+2}(F_{n+1}+F_{n+4}),$$
we obtain
$$K_n=\frac{F_{n+3}^2-F_{n+1}F_{n+2}}{F_{n+1}F_{n+2}F_{n+3}}=\frac{F_{n+2}F_{n+4}-F_nF_{n+3}}{F_{n+1}F_{n+2}F_{n+3}}=\frac{F_{n+4}}{F_{n+1}F_{n+3}}-\frac{F_n}{F_{n+1}F_{n+2}}. \tag{1}$$

(i) From (1), we deduce
$$\sum_{n=1}^{\infty}\frac{K_n}{F_nF_{n+4}}=\sum_{n=1}^{\infty}\left(\frac{1}{F_nF_{n+1}F_{n+3}}-\frac{1}{F_{n+1}F_{n+2}F_{n+4}}\right)=\lim_{N\to\infty}\left(\frac{1}{F_1F_2F_4}-\frac{1}{F_{N+1}F_{N+2}F_{N+4}}\right),$$

and therefore,

$$\sum_{n=1}^{\infty} \frac{K_n}{F_n F_{n+4}} = \frac{1}{F_1 F_2 F_4} = \frac{1}{3}.$$

(ii) The required sum is $\sum_{n=1}^{\infty} U_n$, where

$$U_n = \frac{(-1)^n K_n}{F_n F_{n+4}} = \frac{(-1)^n}{F_n F_{n+1} F_{n+3}} + \frac{(-1)^{n+1}}{F_{n+1} F_{n+2} F_{n+4}}.$$

Using Cassini's identity, we calculate

$$\begin{aligned}
U_n &= \frac{F_{n+1}F_{n+3} - F_{n+2}^2}{F_n F_{n+1} F_{n+3}} + \frac{F_{n+2}F_{n+4} - F_{n+3}^2}{F_{n+1} F_{n+2} F_{n+4}} \\
&= \frac{1}{F_n} - \frac{F_{n+2}^2}{F_n F_{n+1} F_{n+3}} + \frac{1}{F_{n+1}} - \frac{F_{n+3}^2}{F_{n+1} F_{n+2} F_{n+4}} \\
&= \frac{F_{n+2}}{F_n F_{n+1}} - \frac{F_{n+2}^2}{F_n F_{n+1} F_{n+3}} - \frac{F_{n+3}^2}{F_{n+1} F_{n+2} F_{n+4}} \\
&= \frac{F_{n+2}}{F_n F_{n+3}} - \frac{F_{n+3}^2}{F_{n+1} F_{n+2} F_{n+4}} \\
&= \frac{F_{n+2}}{F_n F_{n+3}} - \frac{F_{n+3}}{F_{n+1} F_{n+4}} - \frac{F_{n+3}}{F_{n+1} F_{n+4}} \left(\frac{F_{n+3}}{F_{n+2}} - 1 \right) \\
&= \frac{F_{n+2}}{F_n F_{n+3}} - \frac{F_{n+3}}{F_{n+1} F_{n+4}} - \left(\frac{1}{F_{n+2}} - \frac{1}{F_{n+4}} \right),
\end{aligned}$$

so that

$$\sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} \left[\left(\frac{F_{n+2}}{F_n F_{n+3}} - \frac{F_{n+3}}{F_{n+1} F_{n+4}} \right) - \left(\frac{1}{F_{n+2}} - \frac{1}{F_{n+4}} \right) \right] = \frac{F_3}{F_1 F_4} - \frac{1}{F_3} - \frac{1}{F_4} = -\frac{1}{6}.$$

(iii) Again with (1), we obtain

$$\frac{K_n}{F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4}} = \frac{1}{F_n F_{n+1}^2 F_{n+2} F_{n+3}^2} - \frac{1}{F_{n+1}^2 F_{n+2}^2 F_{n+3} F_{n+4}} = V_n + W_n - W_{n+1},$$

where

$$V_n = \frac{1}{F_n F_{n+1}^2 F_{n+2} F_{n+3}^2} - \frac{1}{F_n^2 F_{n+1}^2 F_{n+2} F_{n+3}} \quad \text{and} \quad W_n = \frac{1}{F_n^2 F_{n+1}^2 F_{n+2} F_{n+3}}.$$

Using $F_{n+3} - F_n = 2F_{n+1}$ and $F_{n+3} + F_n = 2F_{n+2}$, we can transform the expression of V_n as follows:

$$\begin{aligned}
V_n &= \frac{F_n - F_{n+3}}{F_n^2 F_{n+1}^2 F_{n+2} F_{n+3}^2} \\
&= \frac{-2}{F_n^2 F_{n+1} F_{n+2} F_{n+3}^2} \\
&= \frac{-1}{2} \cdot \frac{4F_{n+1}F_{n+2}}{F_n^2 F_{n+1}^2 F_{n+2}^2 F_{n+3}^2} \\
&= \frac{-1}{2} \cdot \frac{F_{n+3}^2 - F_n^2}{F_n^2 F_{n+1}^2 F_{n+2}^2 F_{n+3}^2},
\end{aligned}$$

and hence, $V_n = \frac{-1}{2}(Z_n - Z_{n+1})$, where $Z_n = \frac{1}{F_n^2 F_{n+1}^2 F_{n+2}^2}$. Thus,

$$\sum_{n=1}^{\infty} \frac{K_n}{F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4}} = \frac{-1}{2} \sum_{n=1}^{\infty} (Z_n - Z_{n+1}) + \sum_{n=1}^{\infty} (W_n - W_{n+1})$$
$$= \frac{-1}{2} \cdot \frac{1}{F_1^2 F_2^2 F_3^2} + \frac{1}{F_1^2 F_2^2 F_3 F_4} = -\frac{1}{8} + \frac{1}{6} = \frac{1}{24}.$$

Also solved by Brian Bradie, Dmitry Fleischman, Ángel Plaza, Albert Stadler, Yunyong Zhang, and the proposer.

Editor's remark: Albert Stadler showed by induction the identities

(i) $\displaystyle\sum_{n=1}^{k} \frac{1}{F_n F_{n+4}} \left(\frac{1}{F_{n+1}} + \frac{1}{F_{n+2}} - \frac{1}{F_{n+3}} \right) = \frac{1}{3} - \frac{1}{F_{k+1} F_{k+2} F_{k+4}}$,

(ii) $\displaystyle\sum_{n=1}^{k} \frac{(-1)^n}{F_n F_{n+4}} \left(\frac{1}{F_{n+1}} + \frac{1}{F_{n+2}} - \frac{1}{F_{n+3}} \right) = -\frac{1}{6} + \frac{(-1)^k}{F_{k+1} F_{k+3} F_{k+4}}$,

(iii) $\displaystyle\sum_{n=1}^{k} \frac{1}{F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4}} \left(\frac{1}{F_{n+1}} + \frac{1}{F_{n+2}} - \frac{1}{F_{n+3}} \right) = \frac{1}{24} - \frac{1}{2F_{k+1} (F_{k+2} F_{k+3})^2 F_{k+4}}$.

H-913 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Let $r \geq 1$ be an odd integer. Prove that there exist rational numbers P_1, Q_1, P_2, and Q_2 such that

$$\sum_{n=1}^{\infty} \frac{(-1)^{\frac{n(r-1)}{2}}}{F_n F_{n+1} F_{n+2} \cdots F_{n+r}} = P_1 \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} + Q_1$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(F_n F_{n+1} F_{n+2} \cdots F_{n+r})^2} = P_2 \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} + Q_2.$$

Solution by the proposer

Let $r \geq s \geq 1$ and let

$$T_r(s) = \sum_{n=1}^{\infty} \frac{(-1)^{sn}}{\prod_{j=n}^{n+s} F_j \prod_{j=n+r-s}^{n+r} F_j} \quad \text{and} \quad C_r(s) = \frac{1}{F_s F_{r-2s} \prod_{j=1}^{s} F_j \prod_{j=r-s+1}^{r} F_j}.$$

We use the identities

(i) $F_{a+b} + (-1)^b F_{a-b} = F_a L_b$ (see [1](15a));

(ii) $F_{n+a} F_{n+b} = F_n F_{n+a+b} + (-1)^n F_a F_b$ (see [2](20a)).

Since

$$\begin{aligned} L_s F_{n+s} F_{n+r-s} - F_{n+s} F_{n+r} &= F_{n+s}(F_{n+r-s} L_s - F_{n+r}) \\ &= F_{n+s}(F_{n+r} + (-1)^s F_{n+r-2s} - F_{n+r}) \quad \text{(by (i))} \\ &= (-1)^s F_{n+s} F_{n+r-2s} \\ &= (-1)^s F_n F_{n+r-s} + (-1)^{n+s} F_s F_{r-2s} \quad \text{(by (ii))}, \end{aligned}$$

we have

$$(-1)^{n+s} F_s F_{r-2s} - L_s F_{n+s} F_{n+r-s} = -F_{n+s} F_{n+r} - (-1)^s F_n F_{n+r-s}. \tag{1}$$

We have

$$\sum_{n=1}^{\infty}\left(\frac{(-1)^{sn}F_sF_{r-2s}}{\prod_{j=n}^{n+s}F_j\prod_{j=n+r-s}^{n+r}F_j}-\frac{(-1)^{(s-1)n-s}L_s}{\prod_{j=n}^{n+s-1}F_j\prod_{j=n+r-s+1}^{n+r}F_j}\right)$$
$$=\sum_{n=1}^{\infty}\frac{(-1)^{(s-1)n-s}\left((-1)^{n+s}F_sF_{r-2s}-L_sF_{n+s}F_{n+r-s}\right)}{\prod_{j=n}^{n+s}F_j\prod_{j=n+r-s}^{n+r}F_j}$$
$$=\sum_{n=1}^{\infty}\frac{(-1)^{(s-1)n-s+1}\left(F_{n+s}F_{n+r}+(-1)^sF_nF_{n+r-s}\right)}{\prod_{j=n}^{n+s}F_j\prod_{j=n+r-s}^{n+r}F_j}\quad\text{(by (1))}$$
$$=\sum_{n=1}^{\infty}\left(\frac{(-1)^{(s-1)(n-1)}}{\prod_{j=n}^{n+s-1}F_j\prod_{j=n+r-s}^{n+r-1}F_j}-\frac{(-1)^{(s-1)n}}{\prod_{j=n+1}^{n+s}F_j\prod_{j=n+r-s+1}^{n+r}F_j}\right)$$
$$=\frac{1}{\prod_{j=1}^{s}F_j\prod_{j=r-s+1}^{r}F_j}.$$

Therefore, we obtain

$$T_r(s)=\frac{(-1)^sL_s}{F_sF_{r-2s}}T_r(s-1)+C_r(s).$$

Using the above identity repeatedly, it turns out that there are rational numbers p and q such that

$$T_r(s)=pT_r(0)+q. \tag{2}$$

Setting $s=\frac{r-1}{2}$ in (2), there are rational numbers p_1 and q_1 such that

$$\sum_{n=1}^{\infty}\frac{(-1)^{\frac{n(r-1)}{2}}}{F_nF_{n+1}F_{n+2}\cdots F_{n+r}}=p_1\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+r}}+q_1.$$

Setting $s=r$ in (2), there are rational numbers p_2 and q_2 such that

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(F_nF_{n+1}F_{n+2}\cdots F_{n+r})^2}=p_2\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+r}}+q_2.$$

In [2], Brousseau showed that there are rational number a and b such that

$$\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+r}}=a\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1}}+b.$$

Therefore, there are rational numbers P_1, Q_1, P_2, and Q_2 such that

$$\sum_{n=1}^{\infty}\frac{(-1)^{\frac{n(r-1)}{2}}}{F_nF_{n+1}F_{n+2}\cdots F_{n+r}}=P_1\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1}}+Q_1$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(F_nF_{n+1}F_{n+2}\cdots F_{n+r})^2}=P_2\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1}}+Q_2.$$

Example: For $r=3$, we have

(i) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{F_nF_{n+1}F_{n+2}F_{n+3}}=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1}}+\frac{3}{4};$

(ii) $$\sum_{n=1}^{\infty} \frac{(-1)^n}{(F_nF_{n+1}F_{n+2}F_{n+3})^2} = \frac{3}{2}\sum_{n=1}^{\infty} \frac{1}{F_nF_{n+1}} - \frac{43}{16}.$$

References

[1] B. A. Brousseau, *Summation of infinite Fibonacci series*, The Fibonacci Quarterly, **7.2** (1969), 143–168.

[2] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Dover, 2008.

No other solution for this problem was submitted.

H-914 Proposed by Benjamin Lee Warren, New York

Let $O_n = \frac{1}{3}n(2n^2+1)$ denote the nth Octahedral number and $C_n = \frac{1}{6}(n^3+5n+6)$ denote the nth Cake number. Prove the identity

$$C_{F_{2n}} + O_{F_{2n+1}} = C_{F_{2n+2}}.$$

Solution by Steve Edwards, Roswell, GA

We start with $F_{2n+1}^2 - 1 = F_{2n}F_{2n+2}$, which comes from Cassini's formula and can be found in [1]. Multiply both sides by $3F_{2n+1}$ to get

$$3F_{2n+1}(F_{2n+1}^2 - 1) = 3F_{2n}F_{2n+1}F_{2n+2}.$$

Next, use the identity $3F_kF_{k+1}F_{k+2} = F_{k+2}^3 - F_{k+1}^3 - F_k^3$ (also in [1]) to get

$$3F_{2n+1}^3 - 3F_{2n+1} = F_{2n+2}^3 - F_{2n+1}^3 - F_{2n}^3.$$

Equivalently,

$$4F_{2n+1}^3 - 3F_{2n+1} + F_{2n}^3 = F_{2n+2}^3,$$

or

$$4F_{2n+1}^3 + 2F_{2n+1} + F_{2n}^3 = F_{2n+2}^3 + 5F_{2n+1}.$$

Using the defining recurrence, we have $5F_{2n+1} = 5F_{2n+2} - 5F_{2n}$, which gives

$$F_{2n}^3 + 5F_{2n} + 6 + 2F_{2n+1}(2F_{2n+1}^2 + 1) = F_{2n+2}^3 + 5F_{2n+2} + 6,$$

or

$$\frac{1}{6}(F_{2n}^3 + 5F_{2n} + 6) + \frac{1}{3}F_{2n+1}(2F_{2n+1}^2 + 1) = \frac{1}{6}(F_{2n+2}^3 + 5F_{2n+2} + 6).$$

The result follows.

Reference

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, 2nd ed., John Wiley and Sons, 2018.

Also solved by Michel Bataille, Brian Bradie, Charles K. Cook, Dmitry Fleischman, Ralph P. Grimaldi, Won Kyun Jeong, Wei-Kai Lai, Woojun Lee, Hideyuki Ohtsuka, Ángel Plaza, Raphael Schumacher, Jason L. Smith, Albert Stadler, David Terr, Ell Torek, Ándres Ventas, Yunyong Zhang, and the proposer.

H-915 Proposed by the editor

Prove that for all $k, m, n \geq 0$,

$$\sum_{j=0}^{n+2} \binom{n+2}{j} F_{2kj+m} = (L_{2k}+2)\sum_{j=0}^{n} \binom{n}{j} F_{2k(j+1)+m}$$

and
$$\sum_{j=0}^{n+2}\binom{n+2}{j}L_{2kj+m}=(L_{2k}+2)\sum_{j=0}^{n}\binom{n}{j}L_{2k(j+1)+m}.$$

Solution by Jason L. Smith, Decatur, IL

Consider the following application of the binomial theorem:
$$\begin{aligned}\sum_{j=0}^{n+2}\binom{n+2}{j}x^{2kj+m} &= x^m\left(1+x^{2k}\right)^{n+2}\\ &= x^m\left(1+2x^{2k}+x^{4k}\right)\sum_{j=0}^{n}\binom{n}{j}x^{2kj}\\ &= \left(x^{-2k}+2+x^{2k}\right)\sum_{j=0}^{n}\binom{n}{j}x^{2k(j+1)+m}.\end{aligned}$$

Now, let $x=\alpha$. Noting that $\alpha^{-2k}=\beta^{2k}$, we obtain
$$\begin{aligned}A=\sum_{j=0}^{n+2}\binom{n+2}{j}\alpha^{2kj+m} &= \left(\alpha^{2k}+\alpha^{-2k}+2\right)\sum_{j=0}^{n}\binom{n}{j}\alpha^{2k(j+1)+m}\\ &= (L_{2k}+2)\sum_{j=0}^{n}\binom{n}{j}\alpha^{2k(j+1)+m}.\end{aligned}$$

If we let $x=\beta$, we similarly obtain
$$B=\sum_{j=0}^{n+2}\binom{n+2}{j}\beta^{2kj+m}=(L_{2k}+2)\sum_{j=0}^{n}\binom{n}{j}\beta^{2k(j+1)+m}.$$

The desired identities can be obtained by taking $\frac{1}{\sqrt{5}}(A-B)$ and $A+B$.

Also solved by Michel Bataille, Dmitry Fleischman, Ralph P. Grimaldi, Hideyuki Ohtsuka, Ángel Plaza, Albert Stadler, and the proposer.