ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to Robert Frontczak, LBBW, Am Hauptbahnhof 2, 70173 Stuttgart, Germany, or by email at the address robert.frontczak@lbbw.de. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-946 Proposed by Ángel Plaza, Gran Canaria, Spain

For any positive integer k, the k-Fibonacci numbers $F_{k,n}$ and the k-Lucas numbers $L_{k,n}$ satisfy the recurrence relation $u_{n+2} = ku_{n+1} + u_n$ for $n > 0$, with respective initial values $F_{k,0} = 0$, $F_{k,1} = 1$, and $L_{k,0} = 2$, $L_{k,1} = k$. Evaluate

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} \sinh^{-1}\left(\frac{1}{(k^2+4)F_{k,n}F_{k,n+1}}\left(L_{k,n+1}\sqrt{2L_{k,2n}} - L_{k,n}\sqrt{2L_{k,2n+2}}\right)\right).$$

H-947 Proposed by David Terr, Coronado, CA

Find all positive integer solutions (n, p, m) to the Diophantine equations

$$3^n L_p + 4^n = 5^m \qquad \text{and} \qquad 3^n + 4^n L_p = 5^m.$$

H-948 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Let $p > 0$ and $r = 1 + 2p^{-2}$. The sequence $\{U_n\}$ is defined by

$$U_0 = 0,\ U_1 = 1, \quad \text{and} \quad U_{n+2} = pU_{n+1} + U_n \text{ for } n \geq 0.$$

Prove that

$$\sum_{n=1}^{\infty} \tan^{-1}\frac{1}{rU_n^2} = \frac{\pi}{4}.$$

H-949 Proposed by Michel Bataille, Rouen, France

Let r, s be integers with $r \neq 0$ and let n be a positive integer. Prove that

$$F_{(n+1)r+s} = \sum_{k\geq 0} (-1)^{k(r+1)} L_r^{n-2k-1}\left(\binom{n-k}{k}F_{r+s}L_r + (-1)^{r+1}\binom{n-1-k}{k}F_s\right).$$

H-950 Proposed by Kunle Adegoke, Ile-Ife, Nigeria, and the editor

Let $G_n = G_n(a,b)$ be a gibonacci sequence, that is,

$$G_0 = a,\ G_1 = b;\ G_n(a,b) = G_{n-1}(a,b) + G_{n-2}(a,b), \qquad (n \geq 2),$$

where a and b are arbitrary numbers (usually integers) not both zero. For $q \in \mathbb{C}$, prove that

$$\sum_{j=0}^{n} q^j F_j G_{j+k} = \frac{G_{k+1} + G_{k-1}}{5} \sum_{j=0}^{n} F_{2j-1} q^j + \frac{G_{k+2} + G_k}{5} \sum_{j=0}^{n} F_{2j} q^j - \frac{G_{k+1} + G_{k-1}}{5} f(q),$$

or equivalently

$$\sum_{j=0}^{n} q^j F_j G_{j+k} = \frac{G_{k+1}}{5} \sum_{j=0}^{n} L_{2j+1} q^j - \frac{G_{k-1}}{5} \sum_{j=0}^{n} L_{2j-1} q^j - \frac{G_{k+1} + G_{k-1}}{5} f(q),$$

where $f(q)$ equals

$$f(q) = \begin{cases} \frac{(-1)^n q^{n+1} + 1}{q+1}, & \text{if } q \neq -1; \\ n+1, & \text{if } q = -1. \end{cases}$$

SOLUTIONS

H-916 Proposed by Hideyuki Ohtsuka, Saitama, Japan
(Vol. 61, No. 2, May 2023)

Let r and s be positive odd integers. Prove that

$$\prod_{n=1}^{\infty} \frac{F_{2n-1} + F_r}{F_{2n-1} + F_s} = \alpha^{\frac{r^2 - s^2}{4}}.$$

Solution by Albert Stadler, Herrliberg, Switzerland

The problem statement holds trivially true for $r = s$, and if it holds true for $r > s$, then it holds true for $r < s$ (by taking reciprocals). So, we may assume that $r > s$. Clearly,

$$\left(x^a + \frac{1}{x^a} + x^b + \frac{1}{x^b}\right) = \left(x^a + x^b\right)\left(1 + x^{-a-b}\right).$$

So

$$\begin{aligned}
\prod_{n=1}^{\infty} \frac{F_{2n-1} + F_r}{F_{2n-1} + F_s} &= \prod_{n=1}^{\infty} \left(\frac{\alpha^{2n-1} + \frac{1}{\alpha^{2n-1}} + \alpha^r + \frac{1}{\alpha^r}}{\alpha^{2n-1} + \frac{1}{\alpha^{2n-1}} + \alpha^s + \frac{1}{\alpha^s}} \right) \\
&= \prod_{n=1}^{\infty} \left(\frac{1 + \alpha^{-2n+1+r}}{1 + \alpha^{-2n+1+s}} \right) \prod_{n=1}^{\infty} \left(\frac{1 + \alpha^{-2n+1-r}}{1 + \alpha^{-2n+1-s}} \right) \\
&= \prod_{n=1}^{\frac{r-s}{2}} \frac{1 + \alpha^{-2n+1+r}}{1 + \alpha^{-2n+1-s}} \\
&= \frac{\left(1 + \alpha^{r-1}\right)\left(1 + \alpha^{r-3}\right) \cdots \left(1 + \alpha^{s+1}\right)}{\left(1 + \alpha^{-s-1}\right)\left(1 + \alpha^{-s-3}\right) \cdots \left(1 + \alpha^{-r+1}\right)} \\
&= \alpha^{((s+1)+(s+3)+\cdots+(r-1))} = \alpha^{\frac{r^2 - s^2}{4}}.
\end{aligned}$$

Also solved by Michel Bataille, Steve Edwards, Dmitry Fleischman, Won Kyun Jeong, Ángel Plaza, Yunyong Zhang, and the proposer.

Editor's remark: Steve Edwards observed that the corresponding product with Lucas numbers equals $K\alpha^{\frac{r^2-s^2}{4}}$, where K depends on the values modulo 4 of r and s. If $r \equiv 1$ and $s \equiv 3$, then

$$K = \frac{1}{\sqrt{5}} \frac{L_{\frac{1-r}{2}}}{F_{\frac{1+s}{2}}} \cdot \frac{F_{\frac{3-r}{2}}}{L_{\frac{3+s}{2}}} \cdots \frac{L_{-\frac{1+s}{2}}}{F_{-\frac{1-r}{2}}}.$$

If $r \equiv 3$ and $s \equiv 1$, then

$$K = \sqrt{5} \frac{F_{\frac{1-r}{2}}}{L_{\frac{1+s}{2}}} \cdot \frac{L_{\frac{3-r}{2}}}{F_{\frac{3+s}{2}}} \cdots \frac{F_{-\frac{1+s}{2}}}{L_{-\frac{1-r}{2}}}.$$

If $r \equiv s \equiv 1$, then

$$K = \frac{L_{\frac{1-r}{2}}}{L_{\frac{1+s}{2}}} \cdot \frac{F_{\frac{3-r}{2}}}{F_{\frac{3+s}{2}}} \cdots \frac{F_{-\frac{1+s}{2}}}{F_{-\frac{1-r}{2}}}.$$

Finally, if $r \equiv s \equiv 3$, then

$$K = \frac{F_{\frac{1-r}{2}}}{F_{\frac{1+s}{2}}} \cdot \frac{L_{\frac{3-r}{2}}}{L_{\frac{3+s}{2}}} \cdots \frac{L_{-\frac{1+s}{2}}}{L_{-\frac{1-r}{2}}}.$$

H-917 Proposed by Benjamin Lee Warren, New York, NY
(Vol. 61, No. 2, May 2023)

Let $O_n = \frac{1}{3}n(2n^2+1)$ denote the nth Octahedral number. Let $T_n = \frac{1}{6}n(n+1)(n+2)$ denote the nth Tetrahedral number. Then, prove the identity

$$O_{F_{2n}} + T_{F_{2n-1}-1} = T_{F_{2n+1}-1}.$$

First Solution by Raphael Schumacher, ETH Zurich, Switzerland

We deduce from Cassini's identity, $F_{m-1}F_{m+1} - F_m^2 = (-1)^m$, that we have the identity

$$\begin{aligned} F_m^2 + (-1)^m &= F_{m-1}F_{m+1} \\ &= (F_{m+1} - F_m)F_{m+1} \\ &= F_{m+1}^2 - F_m F_{m+1}, \end{aligned}$$

which implies, with $m := 2n$, that

$$3F_{2n}^3 + 3F_{2n} - 3F_{2n}F_{2n+1}^2 + 3F_{2n}^2F_{2n+1} = 3F_{2n}(F_{2n}^2 + 1) - 3F_{2n}(F_{2n+1}^2 - F_{2n}F_{2n+1}) = 0.$$

We deduce from this identity the equation

$$\begin{aligned} F_{2n+1}^3 - F_{2n+1} &= F_{2n+1}^3 - F_{2n+1} + 3F_{2n}^3 + 3F_{2n} - 3F_{2n}F_{2n+1}^2 + 3F_{2n}^2F_{2n+1} \\ &= 4F_{2n}^3 + 2F_{2n} + (F_{2n+1} - F_{2n})^3 - (F_{2n+1} - F_{2n}) \\ &= 4F_{2n}^3 + 2F_{2n} + F_{2n-1}^3 - F_{2n-1} \end{aligned}$$

and we can compute using the above formula that

$$\begin{aligned}O_{F_{2n}}+T_{F_{2n-1}-1}&=\frac{1}{3}F_{2n}(2F_{2n}^2+1)+\frac{1}{6}(F_{2n-1}-1)F_{2n-1}(F_{2n-1}+1)\\&=\frac{1}{3}(2F_{2n}^3+F_{2n})+\frac{1}{6}(F_{2n-1}^3-F_{2n-1})\\&=\frac{1}{6}(4F_{2n}^3+2F_{2n}+F_{2n-1}^3-F_{2n-1})\\&=\frac{1}{6}(F_{2n+1}^3-F_{2n+1})\\&=\frac{1}{6}(F_{2n+1}-1)F_{2n+1}(F_{2n+1}+1)\\&=T_{F_{2n+1}-1}.\end{aligned}$$

Second Solution by Hans J. H. Tuenter, Toronto, Canada

We shall prove something more general. Define the polynomials $p(t)=2t(2t^2+1)$ and $q(t)=t(t^2-1)$. Proving the identity is the same as showing that $p(F_{2n})+q(F_{2n-1})=q(F_{2n+1})$. Let μ_n be defined by the recurrence $\mu_{n+2}=x\mu_{n+1}+\mu_n$, with arbitrary, initial conditions μ_0 and μ_1. It is tedious, but not difficult, to verify that

$$x(x^2+3)p(\mu_n)+4(q(\mu_{n-1})-q(\mu_{n+1}))=2x\mu_n(5+x^2+6\,[\mu_n^2-\mu_{n-1}\mu_{n+1}]).$$

The term in square brackets, $c_n=\mu_n^2-\mu_{n-1}\mu_{n+1}$, obeys the trivial recurrence $c_n=-c_{n-1}$, and results in $c_n=(-1)^{n-1}c_1$. This then gives the general identity

$$x(x^2+3)p(\mu_n)+4q(\mu_{n-1})=4q(\mu_{n+1})+2x\mu_n(5+x^2-6(-1)^n[\mu_1^2-\mu_0\mu_2]).$$

Translating this back to the original question, by using an appropriate choice of parameters, gives the special cases for the Fibonacci and Lucas numbers as

$$O_{F_n}+T_{F_{n-1}-1}=T_{F_{n+1}-1}+\delta_{n\text{ is odd}}F_n \quad\text{and}\quad O_{L_n}+T_{L_{n-1}-1}=T_{L_{n+1}-1}+(3-5\delta_{n\text{ is odd}})L_n,$$

where δ_c is the Kronecker delta, which is 1 when the condition c is true and 0 otherwise. When n is even, the first identity reduces to the original identity that we were asked to prove.

Also solved by Michel Bataille, Brian Bradie, Charles K. Cook and Michael R. Bacon (jointly), Kenny Davenport, Steve Edwards, Dmitry Fleischman, Ralph P. Grimaldi, Won Kyun Jeong, Wei-Kai Lai, Hideyuki Ohtsuka, Ángel Plaza, Jason L. Smith, Albert Stadler, David Terr, Ell Torek, Yunyong Zhang, and the proposer.

H-918 Proposed by Andrés Ventas, Santiago de Compostela, Spain (Vol. 61, No. 2, May 2023)

Prove that

$$\sum_{n=0}^{\infty}\Big(\frac{1}{(L_{6n}/2)}\frac{1}{L_{6n+2}}+\frac{1}{L_{6n+2}}\frac{1}{L_{6n+3}+(L_{6n}/2)}+\frac{1}{L_{6n+4}}\frac{1}{L_{6n+3}+(L_{6n}/2)}+\frac{1}{L_{6n+4}}\frac{1}{(L_{6n+6}/2)}\Big)=\frac{1}{\sqrt{5}}.$$

Solution by Hideyuki Ohtsuka, Saitama, Japan

Using $F_aL_b - L_aF_b = 2(-1)^bF_{a-b}$ (see [1](16b)), we have

$$\frac{F_{2n+2}}{L_{2n+2}} - \frac{F_{2n}}{L_{2n}} = \frac{F_{2n+2}L_{2n} - L_{2n+2}F_{2n}}{L_{2n+2}L_{2n}} = \frac{2F_2}{L_{2n+2}L_{2n}} = \frac{2}{L_{2n+2}L_{2n}}.$$

We have

$$\sum_{n=0}^{\infty}\left(\frac{2}{L_{6n}L_{6n+2}} + \frac{2}{L_{6n+2}(2L_{6n+3}+L_{6n})} + \frac{2}{L_{6n+4}(2L_{6n+3}+L_{6n})} + \frac{2}{L_{6n+4}L_{6n+6}}\right)$$

$$= \sum_{n=0}^{\infty}\left(\frac{2}{L_{6n}L_{6n+2}} + \frac{2(L_{6n+4}+L_{6n+2})}{L_{6n+2}L_{6n+4}(2L_{6n+3}+L_{6n})} + \frac{2}{L_{6n+4}L_{6n+6}}\right)$$

$$= \sum_{n=0}^{\infty}\left(\frac{2}{L_{6n}L_{6n+2}} + \frac{2}{L_{6n+2}L_{6n+4}} + \frac{2}{L_{6n+4}L_{6n+6}}\right)$$

(by $2L_{6n+3} + L_{6n} = L_{6n+3} + L_{6n+2} + L_{6n+1} + L_{6n} = L_{6n+4} + L_{6n+2}$)

$$= \sum_{n=0}^{\infty}\left(\frac{2}{L_{2(3n)}L_{2(3n+1)}} + \frac{2}{L_{2(3n+1)}L_{2(3n+2)}} + \frac{2}{L_{2(3n+2)}L_{2(3n+3)}}\right)$$

$$= \sum_{n=0}^{\infty}\frac{2}{L_{2n}L_{2(n+1)}} = \lim_{m\to\infty}\sum_{n=0}^{m}\left(\frac{F_{2(n+1)}}{L_{2(n+1)}} - \frac{F_{2n}}{L_{2n}}\right)$$

$$= \lim_{m\to\infty}\left(\frac{F_{2(m+1)}}{L_{2(m+1)}} - \frac{F_0}{L_0}\right) = \frac{1}{\sqrt{5}}.$$

Reference

[1] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Dover, 2008.

Also solved by Michel Bataille, Brian Bradie, Dmitry Fleischman, Yunyong Zhang, and the proposer.

<u>H-919</u> **Proposed by D.-M. Bătineţu-Giurgiu, Bucharest, Romania, and Neculai Stanciu, Buzău, Romania**
(Vol. 61, No. 2, May 2023)

(a) If $a > 0$, then compute $\lim_{n\to\infty}\sqrt[n]{(2n-1)!!F_n}(\sqrt[n]{a}-1)$.
(b) If $a > 0$ and $(b_n)_{n\geq1}$ is a positive real sequence with $\lim_{n\to\infty} b_{n+1}/(nb_n) = b > 0$, then compute $\lim_{n\to\infty}\sqrt[n]{b_nF_n}(\sqrt[n]{a}-1)$.
(c) Compute $\lim_{n\to\infty} n^2\sqrt[n]{n!F_n}\sin(1/n^3)$.
(d) Compute $\lim_{n\to\infty} n\sqrt[n]{(2n-1)!!F_n}\sin(1/n^2)$.

Solution by Michel Bataille, Rouen, France

We first consider (b):
As $n \to \infty$, we have $\lim\limits_{n\to\infty}\frac{F_{n+1}}{F_n} = \alpha$; hence, $u_n := \frac{b_nF_n}{n^n}$ satisfies

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(\frac{b_{n+1}}{nb_n}\cdot\frac{F_{n+1}}{F_n}\cdot\left(1+\frac{1}{n}\right)^{-n}\cdot\left(1+\frac{1}{n}\right)^{-1}\right) = \frac{b\alpha}{e}.$$

We deduce that $\lim_{n\to\infty} \sqrt[n]{u_n} = \frac{b\alpha}{e}$ as well and therefore, $\sqrt[n]{b_n F_n} \sim \frac{b\alpha n}{e}$.
Since $\sqrt[n]{a} - 1 = e^{(\ln a)/n} - 1 \sim \frac{\ln a}{n}$, we readily obtain

$$\lim_{n\to\infty} \sqrt[n]{b_n F_n}(\sqrt[n]{a} - 1) = \frac{b\alpha \ln a}{e}.$$

(a) The sequence (b_n) defined by $b_n = (2n-1)!!$ satisfies $\lim_{n\to\infty} b_{n+1}/(nb_n) = 2$; hence, (b) gives

$$\lim_{n\to\infty} \sqrt[n]{(2n-1)!!F_n}(\sqrt[n]{a} - 1) = \frac{2\alpha \ln a}{e}.$$

(c) From (b) we deduce $\sqrt[n]{n!F_n} \sim \frac{\alpha n}{e}$; since $n^2 \sin(1/n^3) \sim n^2 \cdot \frac{1}{n^3} = \frac{1}{n}$, we get

$$\lim_{n\to\infty} n^2 \sqrt[n]{n!F_n} \sin(1/n^3) = \frac{\alpha}{e}.$$

(d) Similarly, from $n \sin(1/n^2) \sim \frac{1}{n}$ and $\sqrt[n]{(2n-1)!!F_n} \sim \frac{2\alpha n}{e}$, we deduce that

$$\lim_{n\to\infty} n \sqrt[n]{(2n-1)!!F_n} \sin(1/n^2) = \frac{2\alpha}{e}.$$

Also solved by Brian Bradie, Dmitry Fleischman, Ángel Plaza, and the proposers.

H-920 Proposed by the editor
(Vol. 61, No. 2, May 2023)

For $m \geq 0$, prove that

$$\sum_{k=1}^{\infty}(\zeta(4k) - 1)F_{4k+m} = \frac{F_m}{2} + \frac{L_{m+2}}{5} + \frac{\pi}{4\sqrt{5}}L_{m+1} \tan \frac{\sqrt{5}\pi}{2} - \frac{\pi}{8\sqrt{5}}L_{m+1}A - \frac{\pi}{8}F_{m+1}B$$

and

$$\sum_{k=1}^{\infty}(\zeta(4k) - 1)L_{4k+m} = \frac{L_m}{2} + F_{m+2} + \frac{\sqrt{5}\pi}{4}F_{m+1} \tan \frac{\sqrt{5}\pi}{2} - \frac{\pi}{8}L_{m+1}B - \frac{\sqrt{5}\pi}{8}F_{m+1}A,$$

where

$$A = \coth(\pi\alpha) + \coth(\pi/\alpha) \quad \text{and} \quad B = \coth(\pi\alpha) - \coth(\pi/\alpha),$$

$\alpha = (1+\sqrt{5})/2$, and $\zeta(s) = \sum_{k=1}^{\infty} 1/k^s$, $\Re(s) > 1$ is the Riemann zeta function.

Solution by Brian Bradie, Newport News, VA

Subtracting

$$\sum_{k=1}^{\infty} x^{2k} = \frac{x^2}{1-x^2}$$

from

$$\sum_{k=1}^{\infty} \zeta(2k)x^{2k} = \frac{1}{2}(1 - \pi x \cot \pi x)$$

gives

$$\sum_{k=1}^{\infty}(\zeta(2k) - 1)x^{2k} = \frac{1}{2} - \frac{\pi}{2}x \cot \pi x - \frac{x^2}{1-x^2}.$$

Now, replacing x by ix yields

$$\sum_{k=1}^{\infty}(\zeta(2k) - 1)(-1)^k x^{2k} = \frac{1}{2} - \frac{\pi}{2}x \coth \pi x + \frac{x^2}{1+x^2},$$

and adding these last two expressions and then dividing by 2 produces

$$\sum_{k=1}^{\infty}(\zeta(4k)-1)x^{4k}=\frac{1}{2}-\frac{\pi}{4}x\cot\pi x-\frac{\pi}{4}x\coth\pi x-\frac{x^4}{1-x^4}.$$

It then follows that

$$\sum_{k=1}^{\infty}(\zeta(4k)-1)\alpha^{4k+m}=\frac{\alpha^m}{2}-\frac{\alpha^{m+4}}{1-\alpha^4}-\frac{\pi}{4}\alpha^{m+1}\cot\pi\alpha-\frac{\pi}{4}\alpha^{m+1}\coth\pi\alpha$$

and

$$\sum_{k=1}^{\infty}(\zeta(4k)-1)\beta^{4k+m}=\frac{\beta^m}{2}-\frac{\beta^{m+4}}{1-\beta^4}-\frac{\pi}{4}\beta^{m+1}\cot\pi\beta-\frac{\pi}{4}\beta^{m+1}\coth\pi\beta.$$

From here, note

$$\begin{aligned}
\cot\pi\alpha &= \cot\left(\frac{\pi}{2}+\frac{\pi\sqrt{5}}{2}\right)=-\tan\frac{\pi\sqrt{5}}{2},\\
\cot\pi\beta &= \cot\left(\frac{\pi}{2}-\frac{\pi\sqrt{5}}{2}\right)=\tan\frac{\pi\sqrt{5}}{2},\\
\frac{\alpha^4}{1-\alpha^4} &= -\frac{\alpha^2}{\sqrt{5}},\qquad \frac{\beta^4}{1-\beta^4}=\frac{\beta^2}{\sqrt{5}},\\
\alpha^{m+1} &= \frac{1}{2}(\sqrt{5}F_{m+1}+L_{m+1}),\quad \text{and}\\
\beta^{m+1} &= \frac{1}{2}(L_{m+1}-\sqrt{5}F_{m+1}).
\end{aligned}$$

Thus,

$$\sum_{k=1}^{\infty}(\zeta(4k)-1)\alpha^{4k+m}=\frac{\alpha^m}{2}+\frac{\alpha^{m+2}}{\sqrt{5}}+\frac{\pi}{4}\alpha^{m+1}\tan\frac{\pi\sqrt{5}}{2}-\frac{\pi}{8}(\sqrt{5}F_{m+1}+L_{m+1})\coth\pi\alpha$$

and

$$\sum_{k=1}^{\infty}(\zeta(4k)-1)\beta^{4k+m}=\frac{\beta^m}{2}-\frac{\beta^{m+2}}{\sqrt{5}}-\frac{\pi}{4}\beta^{m+1}\tan\frac{\pi\sqrt{5}}{2}+\frac{\pi}{8}(L_{m+1}-\sqrt{5}F_{m+1})\coth\frac{\pi}{\alpha}.$$

Finally,

$$\begin{aligned}
\sum_{k=1}^{\infty}(\zeta(4k)-1)F_{4k+m} &= \frac{1}{\sqrt{5}}\left(\sum_{k=1}^{\infty}(\zeta(4k)-1)\alpha^{4k+m}-\sum_{k=1}^{\infty}(\zeta(4k)-1)\beta^{4k+m}\right)\\
&= \frac{F_m}{2}+\frac{L_{m+2}}{5}+\frac{\pi}{4\sqrt{5}}L_{m+1}\tan\frac{\pi\sqrt{5}}{2}-\frac{\pi}{8\sqrt{5}}L_{m+1}A-\frac{\pi}{8}F_{m+1}B
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=1}^{\infty}(\zeta(4k)-1)L_{4k+m} &= \sum_{k=1}^{\infty}(\zeta(4k)-1)\alpha^{4k+m}+\sum_{k=1}^{\infty}(\zeta(4k)-1)\beta^{4k+m}\\
&= \frac{L_m}{2}+F_{m+2}+\frac{\pi\sqrt{5}}{4}\tan\frac{\pi\sqrt{5}}{2}-\frac{\pi}{8}L_{m+1}B-\frac{\pi\sqrt{5}}{8}F_{m+1}A,
\end{aligned}$$

where

$$A=\coth(\pi\alpha)+\coth(\pi/\alpha)\quad\text{and}\quad B=\coth(\pi\alpha)-\coth(\pi/\alpha).$$

Also solved by Michel Bataille, Dmitry Fleischman, David Terr, Yunyong Zhang, and the proposer.