RESIDUES OF FIBONACCI-LIKE SEQUENCES

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In the February, 1964, issue of the Fibonacci Quarterly, Brother U. Alfred [1] advanced the conjecture (later proved by J. H. Halton [2]) that, when any Fibonacci number is divided by another Fibonacci number, one or the other of the least positive and negative residues is again a Fibonacci number. The object of this paper is to prove that the only Fibonacci-like sequence for which this is true is the Fibonacci sequence. If zero is excluded as a remainder, then the Lucas sequence has the above property.

The proof falls naturally into two parts. The first part will be to show that every Fibonacci-like sequence, modulo any member of the sequence, is congruent to a sequence made up of a subsequence of the original sequence and the negatives of these values. The second part will be to show that these subsequences are actually remainders of the divisor for only the Fibonacci and Lucas sequences.

Obviously, a sequence has the property described above if and only if any non-zero integral multiple of it does. Since any divisor of two neighboring members of Fibonacci-like sequences divides every member of the sequence, we will consider only sequences with neighboring terms relatively prime. In what follows, H_i will denote the i^{th} member of a general Fibonacci-like sequence defined by $H_{i+1} = H_i + H_{i-1}$, where H_0 and H_1 are arbitrary. The set of integers will be denoted by I, the set of non-negative integers by P, and the set of natural numbers by N.

PART I

Since it is easily established by induction that

$$H_{m+k} = F_k H_{m-1} + F_{k+1} H_m ,$$

for all integers m and k, the following two lemmas readily follow.

Lemma 1: $H_{m+i} \equiv F_i H_{m-1} \pmod{H_m}$ for all integers i.

Lemma 2: $H_{m-i} \equiv F_{-i} H_{m-1} \equiv (-1)^{i+1} H_{m+i} \pmod{H_m}$ for all integers i.

It is known that any number must eventually divide one of the Fibonacci numbers, and that $F_{n-1}^2 = F_{n-2}F_n + (-1)^n$ for all integers n. Applying these results and Lemma 1, it is not difficult to prove Lemmas 3 and 4.

Lemma 3: Let n be any integer such that $F_n \equiv 0 \pmod{H_m}$. Then

$$H_{m\pm n} \equiv 0 \pmod{H_m} .$$

Lemma 4: For the n of Lemma 3, $F_{n-1}^2 \equiv (-1)^n \pmod{H_m}$.

Lemma 5: For the n of Lemma 3, and for all integers i,

$$F_{n-1}H_{m-i} \equiv (-1)^{n+i+1}H_{m-n+i} \pmod{H_m} .$$

Proof: The proof is by induction on i. For i = 0, apply Lemma 3. For i = 1, apply Lemma 1. Assume that Lemma 5 holds for i = k - 1 and i = k - 2, or that

$$F_{n-1}H_{m-(k-1)} \equiv (-1)^{n+k}H_{m-n+(k-1)} \pmod{H_m} ,$$

$$F_{n-1}H_{m-(k-2)} \equiv (-1)^{n+k-1}H_{m-n+(k-2)} \pmod{H_m} .$$

Subtracting the first formula from the second yields the expected result for i = k. Hence, the formula is correct for all $i \in P$. Lemma 2 can be used to extend the result to include negative integers.

Lemma 6: Let t = nq + r. Then, if $q \in N$ and $F_n \equiv 0 \pmod{H_m}$,

$$H_{m-n+t} \equiv F_rF_{n-1}^{q-1}H_{m-1} \pmod{H_m} .$$

Proof: The proof is once again by induction on q. When q = 1, the expression above becomes identical to Lemma 1. Assume that Lemma 6 holds for q = k - 1, or that

$$H_{m-n+t} \equiv F_{t-(k-1)n}F_{n-1}^{k-2}H_{m-1} \pmod{H_m} .$$

But, $F_{t-(k-1)n} = F_{t-kn}F_{n-1} + F_{t-kn+1}F_n \equiv F_{t-kn}F_{n-1} \pmod{H_m}$, since H_m divides F_n by hypothesis. Substituting back into the formula above,

$$H_{m-n+t} \equiv F_{t-kn}F_{n-1}^{k-1}H_{m-1} \pmod{H_m}.$$

Hence, Lemma 6 is true for all $q \in N$.

Theorem 1: For every $i \in I$, there exists a $k \in I$, $m - n \leq k \leq m$, such that

$$H_i \equiv \pm H_k \pmod{H_m},$$

where n is the smallest natural number such that $F_n \equiv 0 \pmod{H_m}$.

Proof: Let $i = m - n + t$, $k = m - n + r$, and $t = nq + r$, $0 \leq r \leq n$. The case $q = 0$ is trivial, since then $t = p$ and $i = k$. The case $q < 0$ is equivalent to $t < 0$. But, by Lemma 2 and properties of congruences,

$$H_{m-n+t} \equiv (-1)^{t+1}H_{m-n+(-t)} \pmod{H_{m-n}} \equiv (-1)^{t+1}H_{m-n+(-t)} \pmod{H_m}.$$

Since $-t > 0$, we need consider only the case $t > 0$ or $q \in N$. By Lemma 6,

$$H_{m-n+t} \equiv F_rF_{n-1}^{q-1}H_{m-1} \pmod{H_m}.$$

By Lemma 1,

$$F_rH_{m-1} \equiv (-1)^{r+1}H_{m-r} \pmod{H_m}.$$

Substituting,

$$H_{m-n+t} \equiv (-1)^{r+1}F_{n-1}^{q-1}H_{m-r} \pmod{H_m}.$$

By Lemma 4,

$$F_{n-1}^2 \equiv (-1)^n \pmod{H_m}.$$

We must now distinguish two cases.

Case 1: If q is odd,

$$F_{n-1}^{q-1} = (-1)^{n(q-1)/2} \pmod{H_m},$$

leading to $H_{m-n+t} = \pm H_{m-r} \pmod{H_m}$, where $m - n \leq m - r \leq m$.

Case 2: If q is even,

$$F_{n-1}^{q-1} = (-1)^{n(q-2)/2} F_{n-1} \pmod{H_m}.$$

By Lemma 5,

$$F_{n-1} H_{m-r} \equiv (-1)^{n+r+1} H_{m-n+r} \pmod{H_m}.$$

Substituting these two results leads to

$$H_{m-n+t} \equiv (-1)^{nq/2} H_{m-n+r} \pmod{H_m}.$$

where $0 \leq r \leq n$, so $m - n \leq m - n + r \leq m$.

In Theorem 1, if H_m divides H_i, we can take $k = m$ or $k = m - n$. While every H_i divides some other member of the sequence (see Lemma 3), it is necessary to notice that zero cannot appear as a member of the subsequence of Theorem 1 unless our Fibonacci-like sequence is the Fibonacci sequence itself. Since zero can occur as a remainder in any Fibonacci-like sequence and since Theorem 1, applied to Fibonacci numbers, leads to the theorem proved by Halton in [2], the only Fibonacci-like sequence which strictly fulfills the requirements of Brother Alfred's conjecture is the Fibonacci sequence.

In Part II, we will investigate Fibonacci-like sequences to determine if any other sequence leaves residues which, in all cases, are either zero or equal in absolute value to members of the original sequence.

PART II

Now, if our sequence is to have the desired property, there must be a set of elements of the sequence whose absolute values are less than that of H_m. The first observation to be made about Fibonacci-like sequences is that far to the right and to the left, the absolute values increase without limit. Hence, we need only examine a small section of the whole sequence to determine if it has the desired property.

There must be at least one H_i with a minimal absolute value, and, because of the divergence of the sequence in both directions, there can be only a finite number of such minima.

Lemma 7: If H_0 is a minimum, $|H_0| \geq 2$, then, if $H_1 > 0$, the only possible remainder equal in absolute value to a member of the original sequence upon division by H_{-2} is $\pm H_0$, and if $H_1 < 0$, the only such remainder for H_2 is $\pm H_0$.

Proof: If H_0 is negative, we will obtain the negative of the sequence for H_0 positive. Hence, consider only $H_0 \geq 2$. None of H_1, H_2, H_{-1}, H_{-2} can be a minima, since each of $|H_i| = H_0 \geq 2$, $i = \pm 1, \pm 2$, leads to a contradiction.

If $H_1 > 0$, to avoid $|H_i| < H_0$ for some i, for the terms near H_0 we can have only the following:

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\begin{aligned}
H_{-3} &= 3H_0 + 2\alpha = H_1 + \alpha \\
H_{-2} &= -(H_0 + \alpha) \\
H_{-1} &= 2H_0 + \alpha \\
H_0 &= H_0 \\
H_1 &= 3H_0 + \alpha \\
H_2 &= 4H_0 + \alpha \\
&\cdots \\
H_i &= L_{i+1}H_0 + F_i\alpha , \qquad \alpha \geq 1 ,
\end{aligned}
$$

where L_n and F_n are respectively the n^{th} Lucas and Fibonacci numbers.

If $H_1 < 0$, with the conditions above we obtain

$$H_i = (-1)^i(L_{i+1}H_0 + F_i\alpha) ,$$

or a new sequence which, except for changes in sign, is the sequence for $H_1 > 0$ reflected about H_0. In particular, $H_2 = -(H_0 + \alpha)$.

Notice that the sequence diverges for $|i| > 2$. From the sequence above, it is easy to see that the only remainder in the sequence for H_{-2} will be $\pm H_0$ when $H_1 > 0$, and when $H_1 < 0$, the only remainder for H_2 will be $\pm H_0$.

Lemma 8: If H_0 is a minimum, $|H_0| \geqq 1$, and neither H_2 nor H_{-2} is a minimum, then the only remainder equal in absolute value to a member of the original sequence upon division by H_{-2} is $\pm H_0$ when $H_1 > 0$, and the only such remainder for H_2 is $\pm H_0$ when $H_1 < 0$.

Proof: Avoiding $|H_2| = |H_0|$ and $|H_{-2}| = |H_0|$ as well as $|H_i| < |H_0|$ leads to the formulae of Lemma 7.

Lemma 9: If H_0 is a minimum, $|H_0| \geqq 2$, then there exist numbers H_i which leave remainders which are neither zero nor equal in absolute value to a member of the original sequence.

Proof: If any number H_j is divided by H_0, the remainder must be less in absolute value than H_0, the minimum of the sequence. Thus, if $|H_0| \geqq 2$, all remainders cannot be zero because any two adjacent terms are relatively prime, and any non-zero remainder is a number not equal in absolute value to a member of the original sequence. So H_0 is a number H_i for the lemma.

Suppose we exclude division by H_0. Since $(H_0, H_1) = 1$, H_1 is not a minimum. Either H_1 is positive or H_1 is negative. Without loss of generality (see proof of Lemma 7), we assume that H_1 is negative. By Theorem 1, if n_2 is the least natural number such that $F_{n_2} \equiv 0 \pmod{H_2}$, and if $t = qn_2 + r$, $0 \leq r < n_2$, for q an odd number,

$$H_{2-n_2+t} \equiv \pm H_{2-r} \pmod{H_2}.$$

Now, $H_{2-r} = H_0$ if and only if $r = 2$. If $|H_0| \geq 2$, $|H_2| \geq 3 = F_4$, so $n_2 \geqq 4$, and at least $0 \leqq r < 4$. Set $t = qn_2 + 3$ for an odd number q, say $q = 1$. Substituting, we have $H_5 \equiv \pm H_{-1} \pmod{H_2}$, and $\pm H_{-1} \not\equiv \pm H_0 \pmod{H_2}$ by inspecting the proof of Lemma 7. Thus, we can take $i = 2$.

Lemma 10: If $|H_0| = 1$ is a minimum, and neither H_2 nor H_{-2} is a minimum, then there exist numbers H_i which leave remainders which are neither zero nor equal in absolute value to a number in the original sequence.

Proof: Without loss of generality, we assume that $H_1 < 0$. If $|H_2| \geq 3$, so that $n_2 \geqq 4$, by Lemma 8 we can use the same proof as for Lemma 9. Since H_2 is not a minimum, $H_2 \neq 1$ and $H_2 \neq -1$. The only remaining case is when $|H_2| = 2$, which leads only to the following sequence,

$$\cdots, -23, 14, -9, 5, -4, 1, -3, -2, -5, -7, -12, -19, -31, -50, \cdots,$$

where $31 \equiv 8 \pmod{-23}$ while neither ± 8 nor ± 15 is in the original sequence.

Theorem 2: The only sequences which possess the property that, upon division by a (non-zero) member of that sequence, the members of the sequence leave least positive or negative residues which are either zero or equal in absolute value to a member of the original sequence are the Fibonacci and Lucas sequences.

Proof: By Lemmas 9 and 10, for a sequence to possess the above property, its minimum must be either $H_0 = 0$ or $|H_0| = 1$ with one of H_2 and H_{-2} also a minimum.

If $H_0 = 0$, we can have only the Fibonacci sequence.

Considering the cases $|H_0| = 1$ and $|H_2| = 1$; $|H_0| = 1$ and $|H_{-2}| = 1$, leads to the Lucas sequence and the negative of the Lucas sequence.

It can be shown that, since when Theorem is applied to Lucas numbers, for each L_k, $|L_k| < |L_m|$ or $L_k = 0 \pmod{L_m}$, that the Lucas numbers do indeed have the property of Theorem 2. The Fibonacci numbers are known to also have this property, as proved by Halton in [2].

We have used a minimum value greater than 2 as a criterion to determine if there exist numbers H_i which leave remainders which do not satisfy Theorem 2. Another criterion is that such numbers H_i exist if and only if $|H_j| \neq |H_{-j}|$ for any j, where the sequence has been renumbered so that either H_0 is the minimum or H_0 is between the two minima H_1 and H_{-1}. This second criterion requires a longer proof, but not a difficult one, done by examining all cases.

Examining several sequences to aid in the formulation of the proofs given here led to an interesting question. If Brother Alfred's conjecture is not true for a whole sequence, can it be true for some elements of the sequence, and if so, which ones?

REFERENCES

1. Brother U. Alfred, "Exploring Fibonacci Residues," Fibonacci Quarterly, Vol. 2, No. 1, Feb., 1964, p. 42.
2. J. H. Halton, "On Fibonacci Residues," Fibonacci Quarterly, Vol. 2, No. 3, Oct., 1964, pp. 217-218.

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