From quantum uncertainty to quantum groups

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Birkhoff-von Neumann talk
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[1] Birkhoff, von Neumann, *The logic of quantum mechanics* (1936).

> *"a calculus of propositions which is formally indistinguishable from the calculus of linear subspaces with respect to set products, linear sums, and orthogonal complements"*

Unfortunately and unfairly, it is a subject of strong criticism.

> *"the tale of quantum logic is not the tale of a promising idea gone bad, it is rather the tale of the unrelenting pursuit of a bad idea"*

> *"a punishment inflicted on nature"*

[2] Woronowicz, *Symétries quantiques* (1995).

A **compact quantum group** is a pair (A, Δ), where

1. A is a unital C^*-algebra,
2. $\Delta \colon A \to A \otimes A$ is a unital $*$-homomorphism,

such that

1. $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$,
2. $\Delta[A] \cdot (A \otimes 1)$ and $\Delta[A] \cdot (1 \otimes A)$ are dense in $A \otimes A$.

One might think

> *“quantum groups seem contrived, or at least misnamed.”*

Another might think

> *“quantum logic seems contrived, or at least misnamed.”*

[3] Heisenberg, *Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik* (1927).

> *"canonically conjugated variables can be determined simultaneously only with a characteristic uncertainty"*

[4] Kennard, *Zur Quantenmechanik einfacher Bewegungstypen* (1927).

$$\sigma_p \sigma_q \geq \hbar/2$$

$$A_1, \ldots, A_n \vdash B$$

When we measure $A_1, \ldots, A_n$, and B,
if $A_1, \ldots, A_n$ are true, then B is true.

Example. $|p| < 1, |q| < 1 \vdash |q| < 1$ is sound.

Example. $|q| < 1, |p| < 1 \vdash |q| < 1$ is not sound.

Units of displacement: mm.

Units of momentum: $m_e \cdot \mathrm{mm/s}$.

Some valid structural rules are obvious:

$$\frac{}{\Gamma, A \vdash A} \qquad \frac{\Gamma \vdash A \quad \Gamma, A \vdash B}{\Gamma \vdash B} \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma, A \vdash B}$$

Some valid structural rules are less obvious:

$$\frac{\Gamma, A, B \vdash A \quad \Gamma, A, B \vdash C \quad \Gamma, B, A \vdash B}{\Gamma, B, A \vdash C}$$

It appears that the validity of structural rules is decidable...

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}$$

$$\frac{\Gamma, A \vdash B \quad \Gamma, \neg A \vdash B}{\Gamma \vdash B} \qquad \frac{\Gamma \vdash \neg A}{\Gamma, A \vdash B}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \qquad \frac{\Gamma \vdash A \rightarrow B}{\Gamma, A \vdash B}$$

If we add these logical rules, then we must have:

I. $A \wedge B = A \cap B$,

II. $\neg A = A^{\perp}$,

III. $A \rightarrow B = (A \cap (A \cap B)^{\perp})^{\perp}$.

[5] K, *A natural deduction system for orthomodular logic* (2023).

Let *SOM* be the sequent calculus with

1. the four displayed structural rules,
2. the seven displayed logical rules.

Theorem. Let A be a logical formula in connectives $\wedge$ and $\neg$. The following are equivalent:

1. $\vdash A$ is derivable in *SOM*,
2. $A \approx \top$ is true in orthomodular lattices.

Question. Which formulas are derivable using all valid structural rules?

$$A(x_1, \ldots, x_n) \in \mathrm{Sub}(\underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n})$$

is a Boolean observable on the quantum systems labeled $x_1, \ldots, x_n$.

[6] Barnum, Caves, Fuchs, Jozsa, Schumacher, *Noncommuting Mixed States Cannot Be Broadcast* (1995).

No variable can occur more than once in the same atomic formula.

A new structural rule:

$$\frac{\Gamma, A, B \vdash C}{\Gamma, B, A \vdash C}$$

when A and B have no free variables in common.

Two new logical rules:

$$\frac{\Gamma \vdash A(y, \overline{z})}{\Gamma \vdash (\forall x) A(x, \overline{z})} \qquad \frac{\Gamma \vdash (\forall x) A(x, \overline{z})}{\Gamma \vdash A(t, \overline{z})}$$

when y is not free in $\Gamma \vdash (\forall x) P$.

[7] Weaver, *Mathematical Quantization* (2001).

If we add these logical rules, then we must have:

IV. $(\forall x) A(x, \overline{z}) = \sup\{B : \mathcal{H} \otimes B \leq A(x, \overline{z})\}$.

$$(\exists x) A(x, \overline{z}) \ :\Leftrightarrow \ \neg(\forall x) \neg A(x, \overline{z})$$

$$A \vee B \ :\Leftrightarrow \ \neg A \wedge \neg B \qquad A \,\&\, B \ :\Leftrightarrow \ \neg(\neg B \to A)$$

[8] Murray, von Neumann, *On rings of operators* (1936).

Each physical system is modelled by a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, or equivalently, a **von Neumann ortholattice** $\mathcal{X} \subseteq \mathrm{Sub}(\mathcal{H})$:

$$\mathcal{X} = \mathrm{Fix}(\mathcal{S}) := \{A \in \mathrm{Sub}(\mathcal{H}) : U[A] = A \text{ for all } U \in \mathcal{S}\}$$

for some group $\mathcal{S} \subseteq \mathrm{Aut}(\mathcal{H})$.

Then, $\mathcal{X}$ is a complete subortholattice of $\mathrm{Sub}(\mathcal{H})$.

A **quantum set** is a von Neumann ortholattice $\mathcal{X} \subseteq \mathrm{Sub}(\mathcal{H})$ such that

every complete Boolean subalgebra is atomic.

Quantum sets model **discrete physical systems**.

[9] K, *Quantum sets* (2020).

Theorem. For each ortholattice $\mathcal{X}$, the following are equivalent:

1. $\mathcal{X}$ is isomorphic to a quantum set,
2. $\mathcal{X}$ is isomorphic to

$$\prod_{i \in I} \mathrm{Sub}(\mathbb{C}^{n_i})$$

for some indexed family $(n_i \in \mathbb{Z}_+ : i \in I)$.

We identify each set S with the quantum set

$$\mathcal{X}(S) := \prod_{s \in S} \mathrm{Sub}(\mathbb{C}).$$

These quantum sets model **discrete classical systems**.

We identify each finite-dimensional Hilbert space $\mathcal{H}$ with the quantum set

$$\mathcal{X}(\mathcal{H}) := \mathrm{Sub}(\mathcal{H}).$$

These quantum sets model **discrete quantum systems**.

Tensor product of quantum sets:

$$\mathcal{X} \subseteq \mathrm{Sub}(\mathcal{H}) \qquad \mathcal{Y} \subseteq \mathrm{Sub}(\mathcal{K})$$

$$\mathcal{X} = \mathrm{Fix}(\mathcal{S}) \qquad \mathcal{Y} = \mathrm{Fix}(\mathcal{T})$$

$$\mathcal{X} \otimes \mathcal{Y} \subseteq \mathrm{Sub}(\mathcal{H} \otimes \mathcal{K})$$

$$\mathcal{X} \otimes \mathcal{Y} := \mathrm{Fix}\{U \otimes V : U \in \mathcal{S},\ V \in \mathcal{T}\}$$

[10] K, *Discrete quantum structures I: Quantum predicate logic* (2024).

Quantum sets provide a semantics for the rules that have been displayed.

$$A(x_1, \ldots, x_n) \in \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n$$

$$\mathcal{X} \subseteq \mathrm{Sub}(\mathcal{H}) \qquad \overline{\mathcal{X}} \subseteq \mathrm{Sub}(\overline{\mathcal{H}})$$

$$\mathcal{X} = \mathrm{Fix}(\mathcal{S}) \qquad \overline{\mathcal{X}} = \mathrm{Fix}(\overline{\mathcal{S}})$$

$$A \in \mathcal{X} \qquad \overline{A} \in \overline{\mathcal{X}}$$

[11] Weaver, *Quantum relations* (2012).

A **relation** $R\colon \mathcal{X} \to \mathcal{Y}$ is an element $R \in \overline{\mathcal{X}} \otimes \mathcal{Y}$.

Quantum sets and relations form a dagger compact closed category **qRel**.

All this structure is <u>definable in quantum predicate logic</u>.

The **identity relation** $I_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{X}$ is the largest element

$$I_{\mathcal{X}} \;\in\; \overline{\mathcal{X}} \otimes \mathcal{X}$$

such that, for all $A \in \mathcal{X}$,

$$I_{\mathcal{X}} \quad \perp \quad \overline{A} \otimes \neg A.$$

Every unit vector in $I_{\mathcal{X}}$ is a Bell state:

When Alice measures $\overline{A}$
and Bob measures A,
they get the same result.

$$x_1 = x_2 \quad :\Leftrightarrow \quad I_{\mathcal{X}}(x_1, x_2)$$

$$\mathcal{X} \xrightarrow{R} \mathcal{Y} \xrightarrow{S} \mathcal{Z}$$

$$(S \circ R)(x, z) \quad :\Leftrightarrow \quad (\exists y_1)(\exists y_2)((R(x, y_1) \wedge S(y_2, z))\,\&\, y_1 = y_2)$$

Example.

$$\mathcal{X} = \mathrm{Sub}(\mathcal{H}) \qquad \mathcal{Y} = \mathrm{Sub}(\mathcal{K}) \qquad \mathcal{Z} = \mathrm{Sub}(\mathcal{L})$$

$$R \in \mathrm{Sub}(\mathcal{B}(\mathcal{H}, \mathcal{K})) \qquad S \in \mathrm{Sub}(\mathcal{B}(\mathcal{K}, \mathcal{L}))$$

$$R = \mathrm{span}(v) \qquad S = \mathrm{span}(w)$$

$$S \circ R = \mathrm{span}(wv)$$

$$\mathcal{X} \xrightarrow{R} \mathcal{Y} \qquad R \in \overline{\mathcal{X}} \otimes \mathcal{Y}$$

$$\mathcal{Y} \xrightarrow{R^\dagger} \mathcal{X} \qquad R^\dagger \in \overline{\mathcal{Y}} \otimes \mathcal{X}$$

$$R^\dagger(y, x) \;:\Leftrightarrow\; \overline{R}(x, y)$$

Theorem. The category **qRel** is

1. symmetric monoidal with products $\mathcal{X} \otimes \mathcal{Y}$ and unit $\mathbf{1} := \mathrm{Sub}(\mathbb{C})$;
2. monoidal closed with dual objects $\overline{\mathcal{X}}$;
3. a dagger category;
4. enriched over complete modular ortholattices.

Furthermore, these structures are all compatible in various ways.

[12] Freyd, Scedrov, *Categories, Allegories* (1990).

A relation $R\colon \mathcal{X} \to \mathcal{Y}$ is a **map** if

1. $R^\dagger \circ R \geq I_{\mathcal{X}}$,
2. $R \circ R^\dagger \leq I_{\mathcal{Y}}$.

Theorem. The following are contravariantly equivalent:

1. the category **qSet**
   - quantum sets,
   - maps in **qRel**,
2. the category **HAW***
   - hereditarily atomic von Neumann algebras,
   - unital normal $*$-homomorphisms.

Let $!_{\mathcal{X}}$ be the maximum relation $\mathcal{X} \to \mathbf{1}$. This is the unique map $\mathcal{X} \to \mathbf{1}$.

[13] K, *Discrete quantum structures II: Examples* (2024).

A **group object** in **qRel** is a quantum set $\mathcal{X}$ with maps

1. $M \colon \mathcal{X} \otimes \mathcal{X} \to \mathcal{X}$,
2. $E \colon \mathbf{1} \to \mathcal{X}$,

such that

1. $M \circ (M \otimes I_{\mathcal{X}}) = M \circ (I_{\mathcal{X}} \otimes M)$,
2. $M \circ (E \otimes I_{\mathcal{X}}) = I_{\mathcal{X}}$,
3. $M \circ (I_{\mathcal{X}} \otimes E) = I_{\mathcal{X}}$,
4. $E^\dagger \circ M \circ (!_{\mathcal{X}}^\dagger \otimes I_{\mathcal{X}}) = !_{\mathcal{X}}$,
5. $E^\dagger \circ M \circ (I_{\mathcal{X}} \otimes !_{\mathcal{X}}^\dagger) = !_{\mathcal{X}}$.

Vaes proved the following (published in [12]):

Theorem. There is a canonical one-to-one correspondence between

1. group objects in **qRel**,
2. discrete quantum groups.

[14] Podleś, Woronowicz, *Quantum deformation of Lorentz group* (1990).

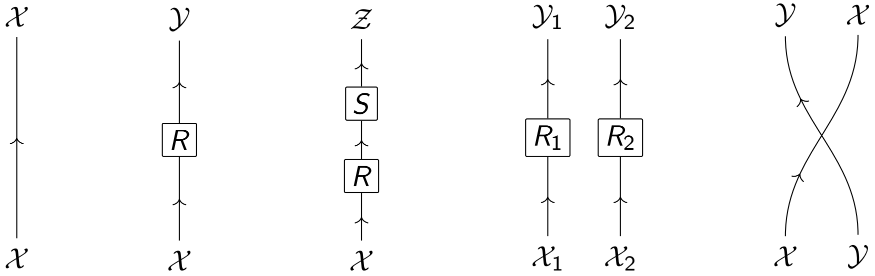
Theorem. There is a canonical one-to-one correspondence between

1. discrete quantum groups,
2. compact quantum groups.

[15] Pontryagin, *The theory of topological commutative groups* (1934).

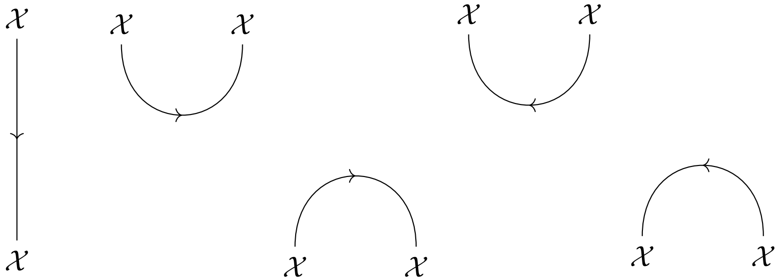
[16] Abramsky, Coecke, *A categorical semantics of quantum protocols* (2004).

The graphical representation of symmetric monoidal structure.



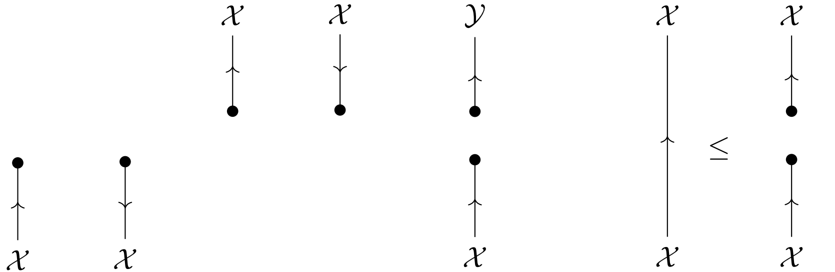
$I_{\mathcal{X}}$ $\quad R \quad$ $S \circ R$ $\quad R_1 \otimes R_2 \quad$ $\beta_{\mathcal{X},\mathcal{Y}}$

The graphical representation of dual objects.



$1_{\overline{\mathcal{X}}}$ $\eta_{\mathcal{X}}$ $\varepsilon_{\mathcal{X}}$ $\varepsilon_{\mathcal{X}}^{\dagger} = \eta_{\overline{\mathcal{X}}}$ $\eta_{\mathcal{X}}^{\dagger} = \varepsilon_{\overline{\mathcal{X}}}$

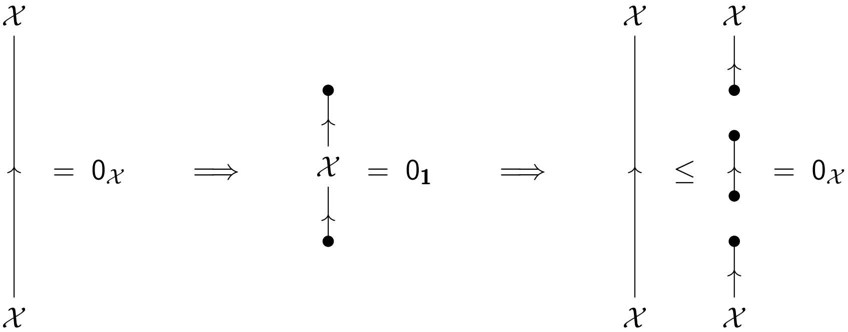
The graphical representation of maximum relations.



$!_{\mathcal{X}}$ $\quad$ $!_{\overline{\mathcal{X}}}$ $\quad$ $!^{\dagger}_{\mathcal{X}}$ $\quad$ $!^{\dagger}_{\overline{\mathcal{X}}}$ $\quad$ $!^{\dagger}_{\mathcal{Y}} \circ !_{\mathcal{X}}$ $\quad$ $I_{\mathcal{X}}$ $\quad$ $!^{\dagger}_{\mathcal{X}} \circ !_{\mathcal{X}}$

Let $0_{\mathcal{X}}$ be the minimum relation $\mathcal{X} \to \mathcal{X}$. This is the zero relation on $\mathcal{X}$.

Characterizations of zero objects in **qRel**:



A **group object** in **qRel** is a triple $(\mathcal{X}, M, E)$ such that