Entropy in multimatrix algebras

Andre Kornell

New Mexico State University

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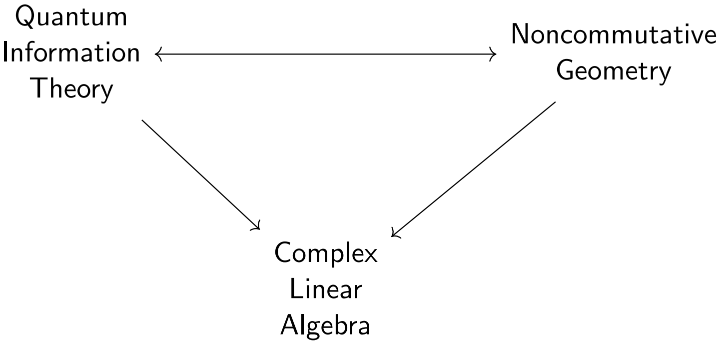

Quantum
Information
Theory
Noncommutative
Geometry
Complex
Linear
Algebra

Definition. A **density matrix** is a Hermitian matrix $\rho \in M_n(\mathbb{C})$ such that

1. each eigenvalue of ρ is nonnegative,
2. $\mathrm{tr}(\rho) = 1$.

Theorem (K). Let $u_1, \ldots, u_k, v_1, \ldots, v_k \in M_{m\times n}(\mathbb{C})$. These are equivalent:

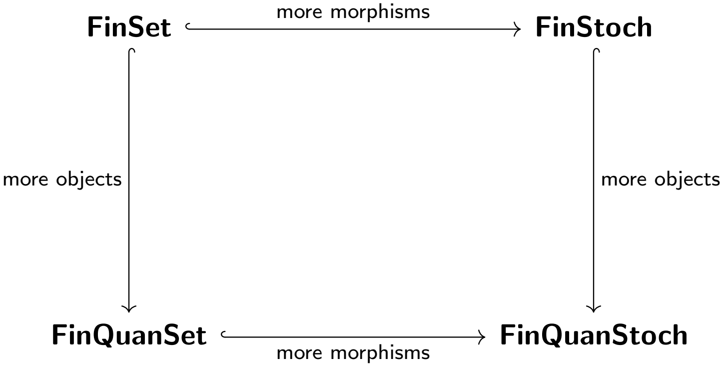
1. $a \mapsto \sum_i v_i^\dagger \cdot a \cdot u_i$ is a unital $*$-homomorphism $M_m(\mathbb{C}) \to M_n(\mathbb{C})$,
2. for each $d \geq 1$ and each density matrix $\rho \in M_{dn}(\mathbb{C})$,

$$\rho' = \sum_i (1_d \otimes u_i) \cdot \rho \cdot (1_d \otimes v_i^\dagger)$$

is a density matrix $\rho' \in M_{dm}(\mathbb{C})$ such that

$$\log n - \mathrm{tr}(\rho \cdot \log \rho) \geq \log m - \mathrm{tr}(\rho' \cdot \log \rho').$$

The setting for this talk consists of four categories:



Our **categories** are all distributive symmetric monoidal categories:

1. monoidal products $X \otimes Y$,
2. binary coproducts $X \oplus Y$,
3. distributors $X \otimes (Y \oplus Z) \cong (X \otimes Y) \oplus (X \otimes Z)$.

Our **functors** preserve $\otimes$ and $\oplus$ up to isomorphism.

Example. The category **FinSet** of finite sets and maps.

Example. The category **Ab** of Abelian groups and group homomorphisms.

Example. The category **FinQuanStoch** of multimatrix algebras and trace-preserving completely positive maps.

Definition. A **multimatrix algebra** is a finite-dim. complex algebra A with an operation $(-)^*\colon A \to A$ such that

1. $(za)^* = \overline{z}a^*$,
2. $(a+b)^* = a^* + b^*$,
3. $(a \cdot b)^* = b^* \cdot a^*$,
4. $a^{**} = a$,
5. $a^* \cdot a = 0 \;\Rightarrow\; a = 0$.

(Our algebras are associative and unital but not necessarily commutative.)

Example. $A = M_n(\mathbb{C})$, with a^* being the conjugate transpose of a.

Proposition. If A is a multimatrix algebra, then

$$A \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_\ell}(\mathbb{C}).$$

Definition. Let A and B be multimatrix algebras.

1. An element $a \in A$ is **positive** if $a = r^2$ for some $r \in A$ with $r^* = r$.
2. A map $\varphi\colon A \to B$ is **positive** if φ is linear and

$$a \text{ is positive } \implies \varphi(a) \text{ is positive.}$$

3. A map $\varphi\colon A \to B$ is **completely positive** if

$$\mathrm{id} \otimes \varphi\colon M_d(A) \to M_d(B)$$

is positive for all $d \geq 1$. (Recall that $M_d(A) = M_d(\mathbb{C}) \otimes A$).

Example. On $M_2(\mathbb{C})$, $a \mapsto a^T$ is positive but not completely positive.

Example. On $M_2(\mathbb{C})$, $a \mapsto vav^*$ is completely positive for each $v \in M_2(\mathbb{C})$.

Definition. For each multimatrix algebra A, let

$$\mathrm{tr}\colon A \to \mathbb{C}$$

be the unique linear map such that

1. $\mathrm{tr}(a \cdot b) = \mathrm{tr}(b \cdot a)$,
2. $\mathrm{tr}(b) = 1$ whenever $b^2 = b = b^*$ and $b \cdot A \cdot b = \mathbb{C}b$.

Example. If $A = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_\ell}(\mathbb{C})$, then

$$\mathrm{tr}\colon (a_1, \ldots, a_\ell) \mapsto \mathrm{tr}(a_1) + \cdots + \mathrm{tr}(a_\ell).$$

Definition. The category **FinQuanStoch**:

1. an object A is a multimatrix algebra,
2. a morphism $\varphi\colon A \to B$ is a completely positive map such that

$$\mathrm{tr}(\varphi(a)) = \mathrm{tr}(a).$$

From the perspective of quantum information theory,

1. an object of **FinQuanStoch** is a finite data type,
2. a morphism of **FinQuanStoch** is a data channel.

Example. The measurement of a qubit is a morphism $M_2(\mathbb{C}) \to \mathbb{C}^2$.

Example. The initialization of two qubits and three bits is a morphism

$$\mathbb{C} \to M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2.$$

$$\textbf{FinStoch} \xhookrightarrow[\text{full}]{} \textbf{FinQuanStoch}$$

Definition. The category **FinStoch**:

1. an object X is a finite set,
2. a morphism $p\colon X \to Y$ is a stochastic map,

$$p(y|x) \in [0,1], \qquad \sum_{y \in Y} p(y|x) = 1.$$

Definition. The inclusion functor:

$$p\colon X \to Y \qquad \varphi_p\colon \mathbb{C}^X \to \mathbb{C}^Y$$

$$\varphi_p(a)(y) = \sum_{x \in X} p(y|x) a(x)$$

Let μ, φ, and ψ be trace-preserving completely positive maps

$$\mu\colon \mathbb{C} \to A \otimes B \qquad \varphi\colon \mathbb{C}^2 \otimes A \to \mathbb{C}^2 \qquad \psi\colon B \otimes \mathbb{C}^2 \to \mathbb{C}^2.$$

The composition

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \xrightarrow{\mathrm{id}\otimes\mu\otimes\mathrm{id}} \mathbb{C}^2 \otimes A \otimes B \otimes \mathbb{C}^2 \xrightarrow{\varphi\otimes\psi} \mathbb{C}^2 \otimes \mathbb{C}^2$$

is essentially a stochastic map $\{0,1\} \times \{0,1\} \to \{0,1\} \times \{0,1\}$.

Definition.

1. If A and B are commutative, this is a **classical correlation**.
2. If A and B are arbitrary, this is a **quantum correlation**.

Theorem (Bell, 1964).

There are quantum correlations that are not classical correlations.

$$\mathbf{FinSet} \xhookrightarrow[\text{same obj.}]{} \mathbf{FinStoch}$$

Definition. The inclusion functor:

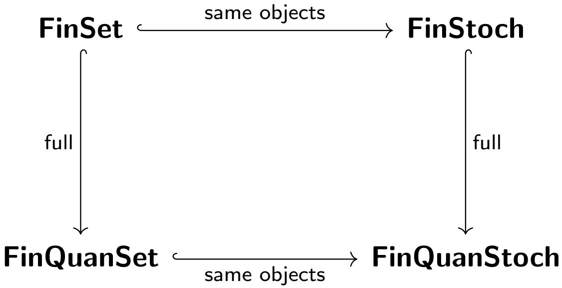
$$f\colon X \to Y \qquad p_f\colon X \to Y \qquad p_f(y|x) = \begin{cases} 1 & y = f(x) \\ 0 & \text{otherwise} \end{cases}$$

Theorem. Let $p\colon X \to Y$ be a stochastic map.

$$p = p_f \text{ for some } f \quad \Longleftrightarrow \quad S(m) \geq S(p \circ m) \text{ for all } m\colon \{\star\} \to X$$

Definition. The **entropy** of m is $S(m) = -\sum_{x \in X} m(x|\star) \log m(x|\star)$.

Can this theorem be generalized from **FinStoch** to **FinQuanStoch**?



What is **FinQuanSet**? The answer depends on perspective.

$$\mathbb{C} \xrightarrow{\text{prepare superposition}} M_2(\mathbb{C}) \xrightarrow{\text{measure}} \mathbb{C}^2$$

This $p\colon \mathbb{C} \to \mathbb{C}^2$ is not **deterministic**, i.e., not of the form $p = p_f$.

Which step introduces randomness?

Definition. The category **MulMat**:

1. an object A is a multimatrix algebra,
2. a morphism $\pi\colon A \to B$ is a unital $*$-homomorphism.

Definition.

$$\mathbf{FinQuanSet} \simeq \mathbf{MulMat}^{\mathrm{op}}$$

This definition is the core of noncommutative geometry.

This definition is comparable to the proposition that

$$\mathbf{AffSch} \simeq \mathbf{CommRing}^{\mathrm{op}}.$$

$$\mathbf{FinSet} \xhookrightarrow[\text{full}]{} \mathbf{FinQuanSet}$$

Definition. The inclusion functor:

$$f\colon X \to Y \qquad \pi_f\colon \mathbb{C}^Y \to \mathbb{C}^X \qquad \pi_f(b) = b \circ f$$

$$\mathbf{FinQuan} \xhookrightarrow[\text{full}]{} \mathbf{FinQuanStoch}$$

Definition. The inclusion functor:

$$\pi\colon B \to A \qquad \varphi_\pi\colon A \to B \qquad \mathrm{tr}(\varphi_\pi(a) \cdot b) = \mathrm{tr}(a \cdot \pi(b))$$

Theorem. Let $p\colon X \to Y$ be a stochastic map.

$$p = p_f \text{ for some } f \quad \Longleftrightarrow \quad S(m) \geq S(p \circ m) \text{ for all } m\colon \{\star\} \to X$$

Question. Let $\varphi\colon A \to B$ be a trace-preserving completely positive map.

$$\varphi = \varphi_\pi \text{ for some } \pi \quad \overset{?}{\Longleftrightarrow} \quad S_{vN}(\mu) \geq S_{vN}(\varphi \circ \mu) \text{ for all } \mu\colon \mathbb{C} \to A$$

Definition. The **von Neumann entropy** of μ is

$$S_{vN}(\mu) = -\mathrm{tr}(\mu(1) \cdot \log \mu(1)).$$

Answer. No. Measurement $\varphi\colon M_2(\mathbb{C}) \to \mathbb{C}^2$ is of the form $\varphi = \varphi_\pi$, but measurement increases von Neumann entropy.

$$
\begin{array}{ccc}
\mathbf{FinSet} & \xhookrightarrow{\ \simeq\ } & \mathbf{CommMulMat}^{\mathrm{op}} \\
{\scriptstyle \text{full}}\big\downarrow & & \big\downarrow{\scriptstyle \text{full}} \\
\mathbf{FinQuanSet} & \xhookrightarrow[\ \simeq\ ]{} & \mathbf{MulMat}^{\mathrm{op}}
\end{array}
$$

The top equivalence is $X \mapsto \mathbb{C}^X$. Intuitively, so is the bottom equivalence.

No explicit construction of **FinQuanSet** fully captures this intuition.

Example. There is a finite quantum set Q such that $\mathbb{C}^Q \cong M_2(\mathbb{C})$.

For each finite set X, we can prove that $\dim \mathbb{C}^X = \mathrm{card}\, X$.

For each finite **quantum** set X, we **define** that $\mathrm{card}\, X = \dim \mathbb{C}^X$.

Example. We have that $\mathrm{card}\, Q = \dim \mathbb{C}^Q = \dim M_2(\mathbb{C}) = 4$.

For each finite set X and each Hermitian $a \in \mathbb{C}^X$, we can prove that

$$a\colon X \to \mathbb{R}, \qquad \mathrm{Ran}(a) = \mathrm{Sp}(a).$$

For each finite **quantum** set X and each Hermitian $a \in \mathbb{C}^X$, we **imagine** that a is a real-valued function on X that has range $\mathrm{Sp}(a)$.

Example. We imagine that $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \in M_2(\mathbb{C}) \cong \mathbb{C}^Q$ is a real-valued function on Q that has range $\{0, 2\}$.

We view each Hermitian $a \in \mathbb{C}^Q \cong M_2(\mathbb{C})$ as a real-valued function on Q.

Question. How do we sum its values **with multiplicity**?

Bad answer. The sum is $\mathrm{tr}(a)$.

Example. We imagine $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ to be a real-valued function with range $\{1\}$. This is the constant function 1 on Q. The sum of its values is $\mathrm{card}\, Q = 4$.

Good answer. The sum is $2\mathrm{tr}(a)$. The map

$$\sigma\colon \mathbb{C}^Q \to \mathbb{C}, \qquad \sigma(a) = 2\mathrm{tr}(a),$$

is the unique linear map such that

1. $\sigma(\pi(a)) = \sigma(a)$ for each automorphism $\pi\colon \mathbb{C}^Q \to \mathbb{C}^Q$,
2. $\sigma(1_Q) = \mathrm{card}\, Q$.

Let X be a finite quantum set. An element $a \in \mathbb{C}^X$ is a **projection** if $a^2 = a = a^*$ or equivalently if a is Hermitian and $\mathrm{Sp}(a) \subseteq \{0, 1\}$.

Definition. We define the linear map $\sigma\colon \mathbb{C}^X \to \mathbb{C}$ by

1. $\sigma(a \cdot b) = \sigma(b \cdot a)$,
2. $\sigma(a) = \dim(\mathbb{C}^X \cdot a)$ for each projection $a \in \mathbb{C}^X$.

In summary, we imagine the following:

The Hermitian elements of $\mathbb{C}^X$ are real-valued functions on X.

The projections in $\mathbb{C}^X$ are $\{0, 1\}$-valued functions on X.

The projections in $\mathbb{C}^X$ correspond to the subsets of X.

The linear map $\sigma\colon \mathbb{C}^X \to \mathbb{C}$ sums values with multiplicity.

Let X be a finite set, and let $m \colon \{\star\} \to X$ be a stochastic map.

$$\begin{aligned} S(m) &= -\sum_{x \in X} m(x|\star) \log m(x|\star) \\ &= -\sum_{x \in X} \varphi_m(1)(x) \log \varphi_m(1)(x) \\ &= -\sum_{x \in X} (\varphi_m(1) \cdot \log \varphi_m(1))(x) \\ &= -\sigma(\varphi_m(1) \cdot \log \varphi_m(1)) \end{aligned}$$

Definition. Let $\mu \colon \mathbb{C} \to A$ be a trace-preserving completely positive map. The **noncommutative entropy** of μ is

$$S_{NCG}(\mu) = -\sigma(\mu(1) \cdot \log \mu(1)).$$

Question. Let $\varphi \colon A \to B$ be a trace-preserving completely positive map.

$$\varphi = \varphi_\pi \text{ for some } \pi \quad \overset{?}{\Longleftrightarrow} \quad S_{NCG}(\mu) \geq S_{NCG}(\varphi \circ \mu) \text{ for all } \mu \colon \mathbb{C} \to A$$

Answer. No. Reset $\varphi \colon M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is not of the form $\varphi = \varphi_\pi$, but reset never increases noncommutative entropy.

Question. What do we do now?

Answer. We follow the slogan

"Everything that must happen must happen completely."

Let A and B be multimatrix algebras, and let

$$\varphi\colon A \to B$$

be a trace-preserving completely positive map.

Definition. The map φ is **entropy-nonincreasing** if

$$S_{NCG}(\mu) \geq S_{NCG}(\varphi \circ \mu)$$

for all trace-preserving completely positive maps $\mu\colon \mathbb{C} \to A$.

Theorem (K). The following are equivalent:

1. $\varphi^*\colon B \to A$ is a unital $*$-homomorphism,
2. $\mathrm{id} \otimes \varphi\colon M_d(A) \to M_d(B)$ is entropy-nonincreasing for all $d \geq 1$.

Proposition. For each morphism $\mu \colon \mathbb{C} \to A$,

$$S_{NCG}(\mu) = S_{vN}(\mu) + E_\mu(\log \zeta_A),$$

where $\zeta_A \in A$ is defined by $\sigma(a) = \mathrm{tr}(a\zeta_A)$ for all $a \in A$.

Proposition. For each morphism $\mu \colon \mathbb{C} \to A$,

$$S_{NCG}(\mu) = -S_{vN}(\mu || \sigma^*) + \log \dim A,$$

where $\sigma^* \colon \mathbb{C} \to A$ is the adjoint of $\sigma \colon A \to \mathbb{C}$.

The morphism σ^* plays the role of the uniform probability distribution.

The noncommutative entropy $S_{NCG}(\mu)$ is maximized when $\mu = \sigma^*$.

$$S_{NCG}(\sigma^*) = \log \dim A$$