On the category of quantum graphs

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Definition. A **graph** is a pair $G = (V_G, e_G)$ such that

1. V_G is a set,
2. e_G is a symmetric relation on V_G.

Definition. A **quantum graph** is a pair $G = (\mathcal{M}_G, \mathcal{R}_G)$ such that

1. $\mathcal{M}_G \subseteq \mathcal{B}(\mathcal{H}_G)$ is a hereditarily atomic von Neumann algebra,
2. $\mathcal{R}_G \subseteq \mathcal{B}(\mathcal{H}_G)$ is an ultraweakly closed subspace such that

$$\mathcal{M}'_G \cdot \mathcal{R}_G \cdot \mathcal{M}'_G \subseteq \mathcal{R}_G, \qquad \mathcal{R}^*_G = \mathcal{R}_G.$$

Definition. A von Neumann algebra $\mathcal{M}$ is **hereditarily atomic** if

$$\mathcal{M} \cong \bigoplus_{\alpha \in I} M_{n_\alpha}(\mathbb{C}).$$

Such von Neumann algebras are the quantum generalization sets in NCG.

Let G and H be quantum graphs.

Definition. A **homomorphism** $G \to H$ is a unital normal $*$-homomorphism

$$\pi \colon \mathcal{M}_H \to \mathcal{M}_G$$

such that

$$k_1 \cdot \mathcal{R}_G \cdot k_2^\dagger \subseteq \mathcal{R}_H$$

for all $k_1, k_2 \in \mathcal{B}(\mathcal{H}_G, \mathcal{H}_H)$ such that $ak_i = k_i\pi(a)$ for all $a \in \mathcal{M}_G$.

This definition generalizes the classical definition when $\mathcal{M} \cong \ell^\infty(X)$.

It "generalizes" the definition of Stahlke when $\mathcal{M} \cong M_n(\mathbb{C})$.

It is probably "equivalent" to the definition of Weaver for arbitrary $\mathcal{M}$.

(Stahlke and Weaver worked with completely positive maps.)

Definition. The symmetric monoidal category **qGph**:

1. an object is a quantum graph $G = (\mathcal{M}_G, \mathcal{R}_G)$,
2. a morphism $G \to H$ is a homomorphism,
3. the monoidal unit is the quantum graph $(\mathbb{C}, 0)$,
4. the monoidal product $G \,\Box\, H$ is defined by $\mathcal{M}_{G \Box H} = \mathcal{M}_G \,\overline{\otimes}\, \mathcal{M}_H$ and

$$\mathcal{R}_{G \Box H} = \overline{\mathcal{R}_G \otimes \mathbb{C} + \mathbb{C} \otimes \mathcal{R}_H}.$$

The monoidal product $G \boxtimes H$ with $\mathcal{R}_{G \boxtimes H} = \mathcal{R}_G \,\overline{\otimes}\, \mathcal{R}_H$ is less well behaved.

We can construct **qGph** from **qRel**.

Definition. The dagger symmetric monoidal category **qRel**:

1. an object $\mathcal{M}$ of **qRel** is a hereditarily atomic von Neumann algebra,
2. a morphism from $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ to $\mathcal{N} \subseteq \mathcal{B}(\mathcal{K})$ is an ultraweakly closed subspace $\mathcal{R} \subseteq \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\mathcal{N}' \cdot \mathcal{R} \cdot \mathcal{M}' \subseteq \mathcal{R}$,
3. the monoidal unit is $\mathbb{C}$,
4. the monoidal product is $\mathcal{M} \mathbin{\overline{\otimes}} \mathcal{N}$,
5. the monoidal product on morphisms is $\mathcal{V} \mathbin{\overline{\otimes}} \mathcal{W}$,
6. the dagger of a morphism $\mathcal{V}$ is $\mathcal{V}^*$.

The category **qRel** has a more explicit equivalent definition.

Definition. A **quantum set** is a family of nonzero fin.-dim. Hilbert spaces,

$$\mathbb{X} = (X_\alpha \mid \alpha \in A).$$

Definition. A **binary relation** $\mathbb{X} \to \mathbb{Y}$ is a family of operator spaces,

$$R = (R_{\alpha\beta} \subseteq \mathcal{B}(X_\alpha, Y_\beta) \mid \alpha \in A, \beta \in B).$$

Definition. If $R\colon \mathbb{X} \to \mathbb{Y}$ and $S\colon \mathbb{Y} \to \mathbb{Z}$, then

$$(S \circ R)_{\alpha\beta} = \sum_{\beta \in B} S_{\beta\gamma} \cdot R_{\alpha\beta}.$$

Definition. The dagger symmetric monoidal category **qRel**:

1. an object $\mathbb{X}$ is a quantum set,
2. a morphism $\mathbb{X} \to \mathbb{Y}$ is a binary relation,
3. the monoidal unit is $\mathbb{1} = (\mathbb{C})$,
4. the monoidal product is $\mathbb{X} \times \mathbb{Y} = (X_\alpha \otimes Y_\beta \mid (\alpha, \beta) \in A \times B)$,
5. the monoidal product on morphisms is
$$R \times S = (R_{\alpha\beta} \otimes S_{\gamma\delta} \mid (\alpha, \gamma) \in A \times C, (\beta, \delta) \in B \times D)$$
6. the dagger of a morphism $R \colon \mathbb{X} \to \mathbb{Y}$ is $R^\dagger \colon \mathbb{Y} \to \mathbb{X}$, where
$$R^\dagger = (R^*_{\alpha\beta} \mid \beta \in B, \alpha \in A).$$

That $(\mathbf{qRel}, \times, \mathbb{1}, \dagger)$ is a dagger symmetric monoidal category means that

1. $(\mathbf{qRel}, \times, \dagger)$ is a symmetric monoidal category
2. $\dagger$ is a contravariant functor on **qRel** with $(R \times S)^\dagger = R^\dagger \times S^\dagger$,
3. the associators, braidings, and unitors all satisfy $R^{-1} = R^\dagger$.

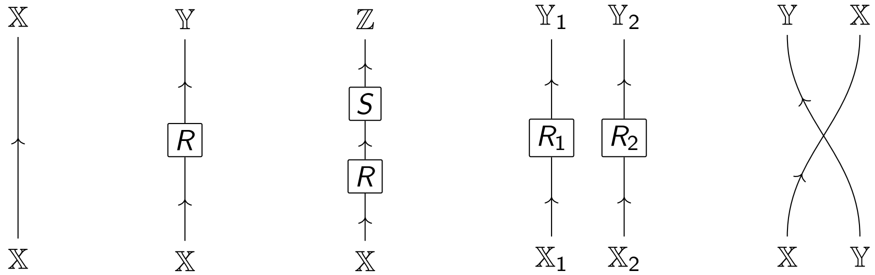
$$\mathbf{Rel} \xhookrightarrow{\text{full dagger symmetric monoidal functor}} \mathbf{qRel}$$

$$A \mapsto \mathbb{A} = (\mathbb{C} \mid \alpha \in A)$$

Definition. The dagger symmetric monoidal category $(\mathbf{Rel}, \times, \{*\}, \dagger)$:

1. **Rel** is the category of sets and binary relations,
2. $A \times B$ is the Cartesian product of A and B,
3. $r^\dagger$ is the converse of r.

The graphical representation of symmetric monoidal structure:



$\mathrm{id}_{\mathbb{X}}$ $\quad R \quad S \circ R \quad R_1 \times R_2 \quad \beta_{\mathbb{X},\mathbb{Y}}$

Proposition. Furthermore, **qRel** is a **dagger compact closed category**: For every object $\mathbb{X}$, there is a "dual" object $\overline{\mathbb{X}}$, and morphisms

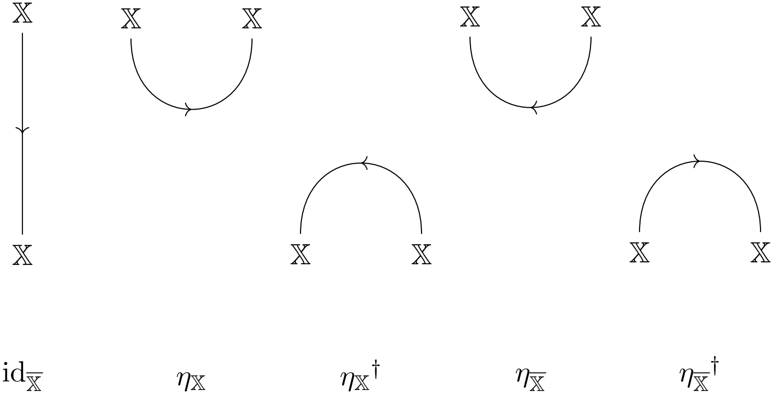
$$\eta_{\mathbb{X}} \colon \mathbb{1} \to \overline{\mathbb{X}} \times \mathbb{X} \qquad \eta_{\overline{\mathbb{X}}} \colon \mathbb{1} \to \mathbb{X} \times \overline{\mathbb{X}}$$

such that

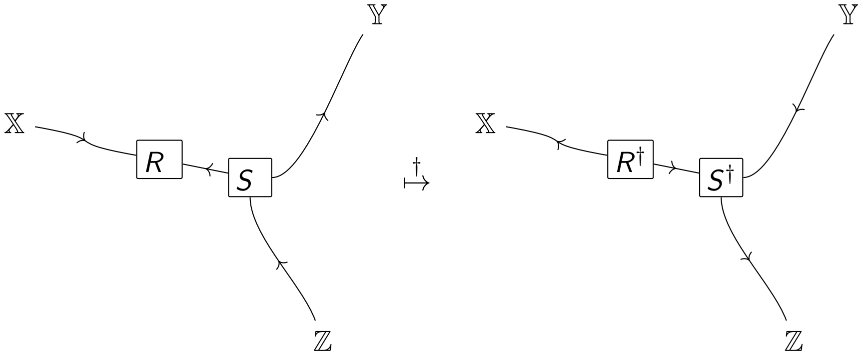
1. $(\mathrm{id}_{\mathbb{X}} \times {\eta_{\mathbb{X}}}^{\dagger}) \circ (\eta_{\overline{\mathbb{X}}} \times \mathrm{id}_{\mathbb{X}}) = \mathrm{id}_{\mathbb{X}}$,
2. $(\mathrm{id}_{\overline{\mathbb{X}}} \times {\eta_{\overline{\mathbb{X}}}}^{\dagger}) \circ (\eta_{\mathbb{X}} \times \mathrm{id}_{\overline{\mathbb{X}}}) = \mathrm{id}_{\overline{\mathbb{X}}}$,
3. $\eta_{\mathbb{X}} = \beta_{\mathbb{X},\overline{\mathbb{X}}} \circ \eta_{\overline{\mathbb{X}}}$.

$$\mathbb{X} = (X_\alpha \mid \alpha \in A) \qquad \Longrightarrow \qquad \overline{\mathbb{X}} = (\overline{X}_\alpha \mid \alpha \in A)$$

The graphical representation of dual objects:



In effect, the domain and codomain of a morphism are subjective.



Proposition. $(\mathbf{qRel}, \times, \mathbb{1}, \dagger)$ is enriched over complete semilattices.

$$R \circ \left(\bigvee_{i \in I} S_i \right) = \bigvee_{i \in I} (R \circ S_i)$$

$$R \times \left(\bigvee_{i \in I} S_i \right) = \bigvee_{i \in I} (R \times S_i) \qquad \left(\bigvee_{i \in I} R_i \right)^\dagger = \bigvee_{i \in I} R_i^\dagger$$

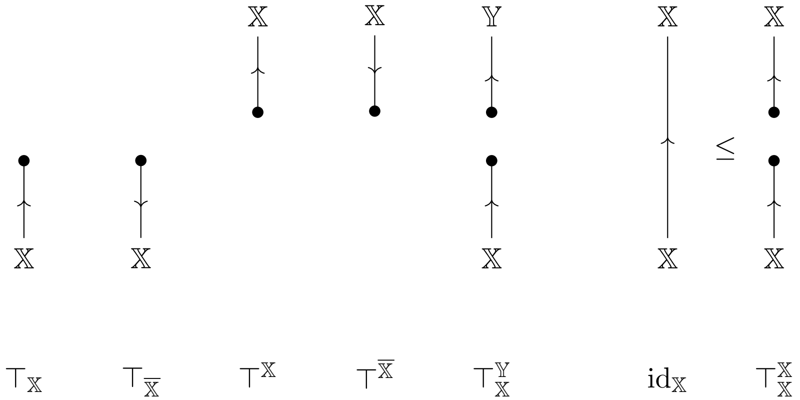
Definition. We define $\top_{\mathbb{X}}^{\mathbb{Y}}$ to be the maximum binary relation $\mathbb{X} \to \mathbb{Y}$.

Proposition. Writing $\top_{\mathbb{X}} = \top_{\mathbb{X}}^{\mathbb{1}}$ and $\top^{\mathbb{Y}} = \top_{\mathbb{1}}^{\mathbb{Y}}$, we have the following:

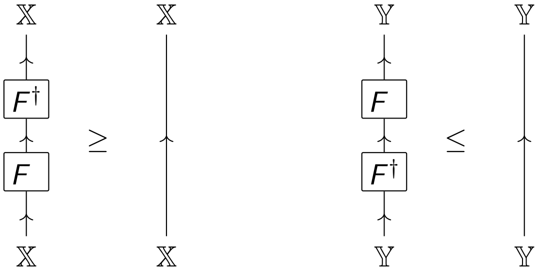
$$(\top_{\mathbb{X}}^{\mathbb{Y}})^\dagger = \top_{\mathbb{Y}}^{\mathbb{X}} \qquad (\top_{\mathbb{X}})^\dagger = \top^{\mathbb{X}} \qquad \top_{\mathbb{X}}^{\mathbb{Y}} = \top^{\mathbb{Y}} \circ \top_{\mathbb{X}}$$

$$\top_{\mathbb{X}} \circ \top^{\mathbb{X}} = 0 \quad \Longleftrightarrow \quad \mathbb{X} = \varnothing$$

The graphical representation of maximum morphisms as loose ends:



Definition. A **map** in **qRel** is a morphism $F\colon X \to Y$ such that

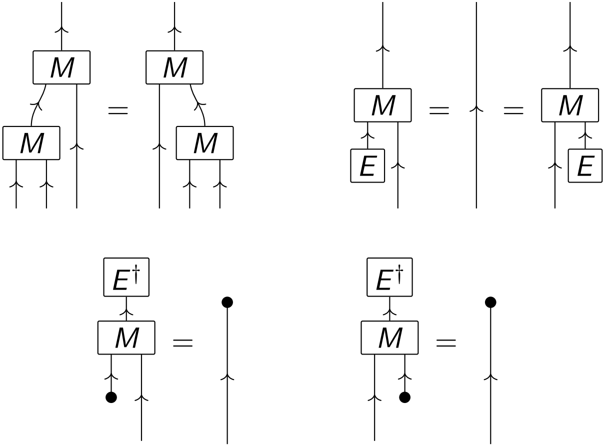


Theorem (K). The subcategory **qSet** of quantum sets and maps is dual to the following symmetric monoidal category:

1. an object is hereditarily atomic von Neumann algebra,
2. a morphism unital normal $*$-homorphism,
3. the monoidal product is the spatial tensor product.

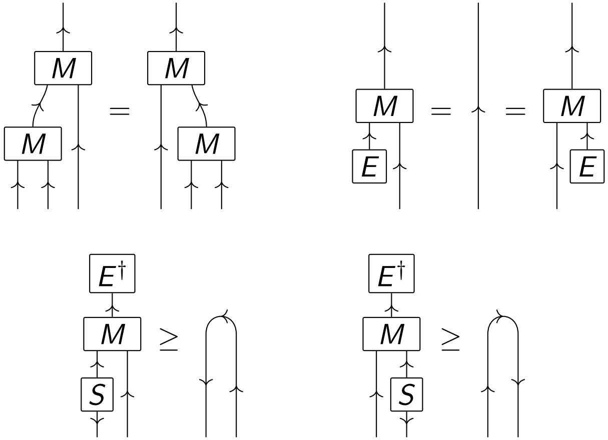
Theorem (Vaes). Up to iso., there is a one-to-one correspondence between

1. discrete quantum groups,
2. quantum sets $\mathbb{X}$ with maps $M\colon \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ and $E\colon \mathbb{1} \to \mathbb{X}$ such that



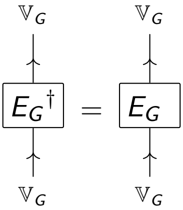
Conjecture. Up to iso., there is a one-to-one correspondence between

1. discrete quantum groups **of Kac type**,
2. quantum sets $\mathbb{X}$ with maps $M\colon \mathbb{X}\times\mathbb{X}\to\mathbb{X}$ and $E\colon \mathbb{1}\to\mathbb{X}$ and $S\colon \overline{\mathbb{X}}\to\mathbb{X}$ such that

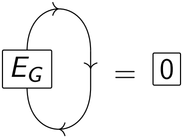


Definition. A **quantum graph** is a pair $G = (\mathbb{V}_G, E_G)$ such that

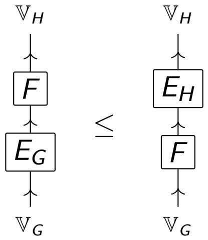
1. $\mathbb{V}_G$ is a quantum set,
2. $E_G \colon \mathbb{V}_G \to \mathbb{V}_G$ is a binary relation satisfying



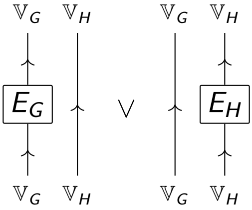
It is **simple** if



Definition. A **homomorphism** $G \to H$ is a map $F\colon \mathbb{V}_G \to \mathbb{V}_H$ such that



Definition. The **box product** $G \,\square\, H$ is defined to be $\mathbb{V}_G \times \mathbb{V}_H$ such that

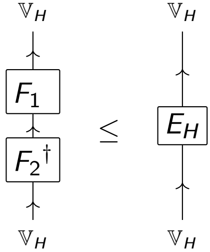


Definition. The symmetric monoidal category **qGph**:

1. an object is a quantum graph $G = (\mathbb{V}_G, E_G)$,
2. a morphism is a homomorphism $G \to H$,
3. the monoidal unit is the quantum graph $\mathbb{1} = (\mathbb{1}, 0_{\mathbb{1}})$,
4. the monoidal product is the box product $G \,\square\, H$.

Proposition. Moreover, **qGph** is enriched over **Gph**, where

$F_1 \sim F_2 \colon G \to H \quad \Longleftrightarrow$



Theorem (K, Lindenhovius). Moreover, **qGph** is closed: there exists

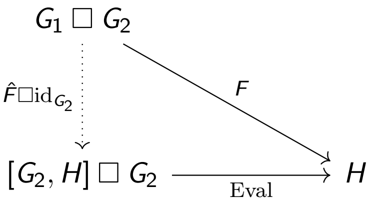
$$[-,-]\colon \mathbf{qGph}^{op} \times \mathbf{qGph} \to \mathbf{qGph}$$

such that, equivalently,

1. there is a natural isomorphism

$$\mathrm{Hom}(G_1 \mathbin{\square} G_2, H) \cong \mathrm{Hom}(G_1, [G_2, H]),$$

2. for all $F\colon G_1 \mathbin{\square} G_2 \to H$, there exists a unique $\hat{F}\colon G_1 \to [G_2, H]$ with



Of course, **qGph** is a quantum generalization of **Gph**.

Definition. The closed symmetric monoidal category **Gph**:

1. an object is a graph (with loops allowed but not muliple edges),
2. a morphism is a homomorphism,
3. the monoidal unit has one vertex and no edges,
4. the monoidal product is the box product, where $(x_1, y_1) \sim (x_2, y_2)$ if

$$(x_1 \sim x_2 \text{ and } y_1 = y_2) \text{ or } (x_1 = x_2 \text{ and } y_1 \sim y_2),$$

5. the internal hom from G to H is $\mathrm{Hom}(G, H)$ with

$$f_1 \sim f_2 \quad \Longleftrightarrow \quad f_1(x) \sim f_2(x) \text{ for all vertices } x \text{ of } G.$$

$$\mathbf{Gph} \xhookrightarrow{\text{full}} \mathbf{qGph}$$

This "inclusion" is a symmetric monoidal functor.

It comes from the dagger symmetric monoidal "inclusion" functor

$$\mathbf{Rel} \xhookrightarrow{\text{full}} \mathbf{qRel}$$

We have an adjunction

$$\mathbf{Gph} \underset{\mathrm{Cls}}{\overset{\hookrightarrow}{\underset{\longleftarrow}{\perp}}} \mathbf{qGph}$$

$$\mathrm{Cls}(G) = \mathrm{Hom}(\mathbb{1}, G)$$

Furthermore, $\mathrm{Cls}[G, H] \cong \mathrm{Hom}(G, H)$.

Definition. A **coloring** of a finite graph G by n colors is a homomorphism

$$G \to K_n.$$

Let $S \subseteq M_d(\mathbb{C})$ be an operator system. Let $S_0 = \{a \in V \mid \mathrm{tr}(a) = 0\}$.

Definition (Paulsen?). A **coloring** of S by n colors is a decomposition

$$\mathbb{C}^d = A_1 \oplus \cdots \oplus A_n$$

such that the compression of S to each subspace A_i consists of the scalars.

Proposition. The colorings of S are in bijection with homomorphisms

$$(\mathbb{C}^d, S_0) \to K_n.$$

Let G be a finite graph, and let K_n be a complete graph of "colors."

Definition. The (G, K_n)-**graph-coloring game** is played by two players

"Alice" and "Bob"

who cooperate using a strategy but cannot communicate with each other.

1. The players are asked about randomly selected vertices x and y of G.
2. The players respond with colors a and b in K_n.
3. Alice sees only x and a, and Bob sees only y and b.
4. The players win if the following conditions both hold:

$$\begin{cases} x = y \quad \Longrightarrow \quad a = b, \\ x \sim y \quad \Longrightarrow \quad a \neq b. \end{cases}$$

The players have a winning strategy iff there is a homomorphism $G \to K_n$.

Proposition. The chromatic number $\chi(G)$ is the least integer n such that there exists a winning strategy for the (G, K_n)-graph-coloring game.

Definition (Avis, Hasegawa, Kikuchi, Sasaki). The **quantum chromatic number** $\chi_q(G)$ is the least integer n such that there exists a winning strategy for the (G, K_n)-coloring-game using entangled quantum systems.

Theorem (Galliard, Wolf). There is a graph G such that $\chi_q(G) < \chi(G)$.

Proposition.

$$\chi(G) \leq n \qquad \Longleftrightarrow \qquad \mathrm{Hom}(G, K_n) \neq \varnothing.$$

Theorem (K, Lindenhovius).

$$\chi_q(G) \leq n \qquad \Longleftrightarrow \qquad [G, K_n] \neq \varnothing.$$

Proof. For all $d \in \mathbb{N}$, write $\mathbb{C}^d = (\mathbb{C}^d, 0_{\mathbb{C}^d})$ and compute that

$$\begin{aligned}
\mathrm{Hom}(\mathbb{C}^d, [G, K_n]) &\cong \mathrm{Hom}(\mathbb{C}^d \,\square\, G, K_n) \\
&\cong \mathrm{Hom}(G \,\square\, \mathbb{C}^d, K_n) \\
&\cong \mathrm{Hom}(G, [\mathbb{C}^d, K_n]) \\
&\cong \mathrm{Hom}(G, \mathrm{Hom}(\mathbb{1}, [\mathbb{C}^d, K_n])) \\
&\cong \mathrm{Hom}(G, \mathrm{Hom}(\mathbb{1} \,\square\, \mathbb{C}^d, K_n)) \\
&\cong \mathrm{Hom}(G, \mathrm{Hom}(\mathbb{C}^d, K_n))
\end{aligned}$$

Thus, $[G, K_n] \neq \varnothing$ iff there exists a hom. $G \to \mathrm{Hom}(\mathbb{C}^d, K_n)$ for some d.

By a theorem of Mančinska and Roberson, this is equivalent to the existence of a winning strategy that uses entangled quantum systems. □

We can explicitly compute $\mathrm{Hom}(\mathbb{C}^d, K_n)$. Let $V_{K_n} = \{1, \dots, n\}$.

Homomorphisms $\mathbb{C}^d \to K_n$ are decompositions $\mathbb{C}^d = A_1 \oplus \cdots \oplus A_n$.

Decompositions $\mathbb{C}^d = A_1 \oplus \cdots \oplus A_n$ and $\mathbb{C}^d = B_1 \oplus \cdots \oplus B_n$ are

$$\text{adjacent} \qquad \Longleftrightarrow \qquad A_i \perp B_i \text{ for all } i.$$

Theorem (Mančinska, Roberson). Let $n \in \mathbb{N}$. The following are equivalent:

1. there exists a quantum winning strategy for the (G, K_n)-g.-c. game,
2. there exists a homomorphism $G \to \mathrm{Hom}(\mathbb{C}^d, K_n)$ for some $d \in \mathbb{N}$.

They proved this theorem for **graph-homomorphism games**, where K_n is replaced by an arbitrary finite graph H. Everything works the same way!