A note about power series asymptotics

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Suppose we have a power series

\[ f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} L(k), \]

where \( L \) is a function satisfying

\[ \lim_{x \to \infty} \frac{L(x)}{L(x + 1)} = 1. \]

If we want to determine the behavior of \( f(x) \) as \( x \to \infty \), one way to proceed would be to determine which terms of the series contribute most to its size. If there is a peak term, it would occur approximately when

\[ \frac{x^k}{k!} L(k) \approx \frac{x^{k+1}}{(k+1)!} L(k+1), \]

which is the same as

\[ \frac{L(k)}{L(k+1)} \approx \frac{x}{k+1}. \]

But if \( k \) is large (this will be true for large \( x \)), then \( L(k)/L(k+1) \approx 1 \), so we have

\[ k \approx x - 1 \approx x. \]

We thus expect that we can approximate the sum by approximating the slowly-varying factors of the summand near this peak at \( k \approx x \):

\[ f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} L(k) \approx \sum_{k=0}^{\infty} \frac{x^k}{k!} L(x) = L(x)e^x. \]

In particular, this heuristic holds when \( L(x) = \log x \) and \( L(x) = 1/\sqrt{x} \), as in these two questions:

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