Spanning Trees and Graphs Embedded in Surfaces

Atlantic Graph Theory Seminar, 12th Jan 2022

Iain Moffatt

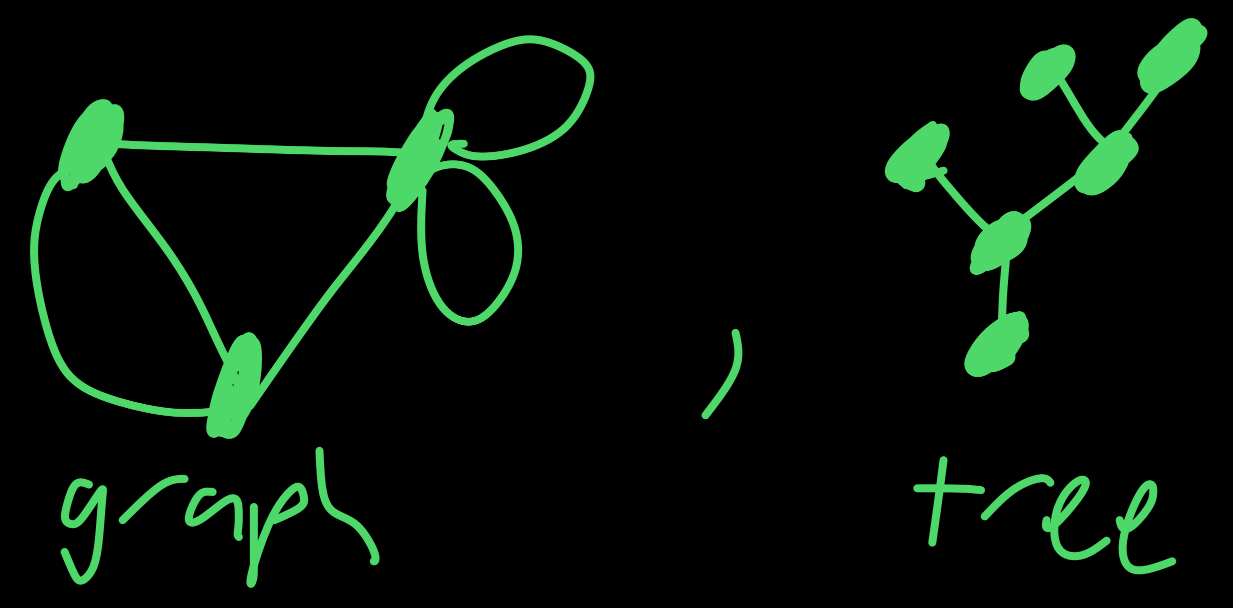
- This is a gentle introduction to delta-matroids via graph theory and topological graph theory

- Loosely based on the expository article: I. Moffatt, *From matrix pivots to graphs in surfaces: touring combinatorics guided by partial duals*, in European Congress of Mathematics Portorož, 20–26 June, 2021

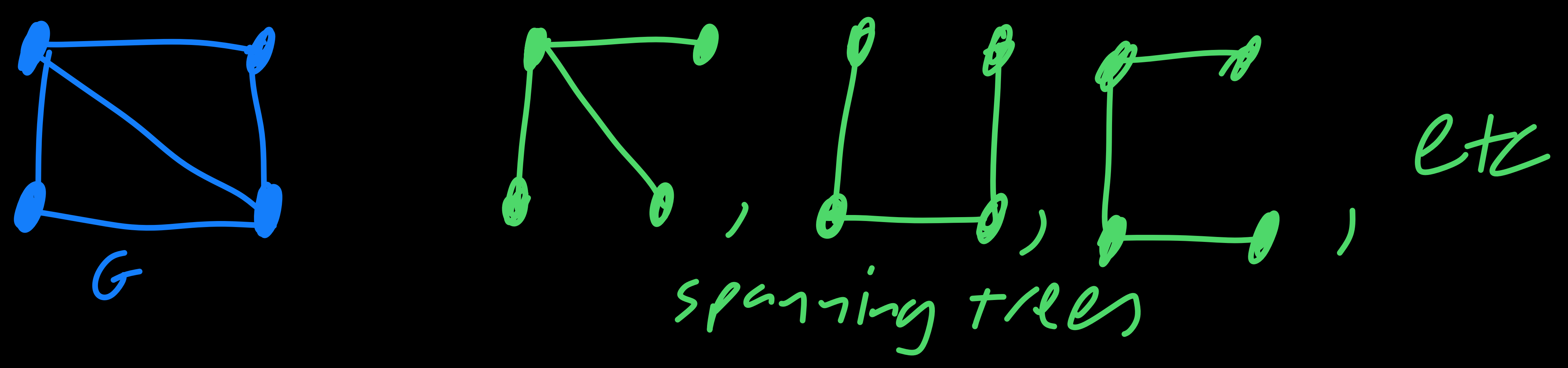
1. Graphs and their spanning trees

A classical question about trees

- Graph = connected multigraph
- tree = connected and no cycles



- spanning tree of G = subgraph + tree + all vertices of G

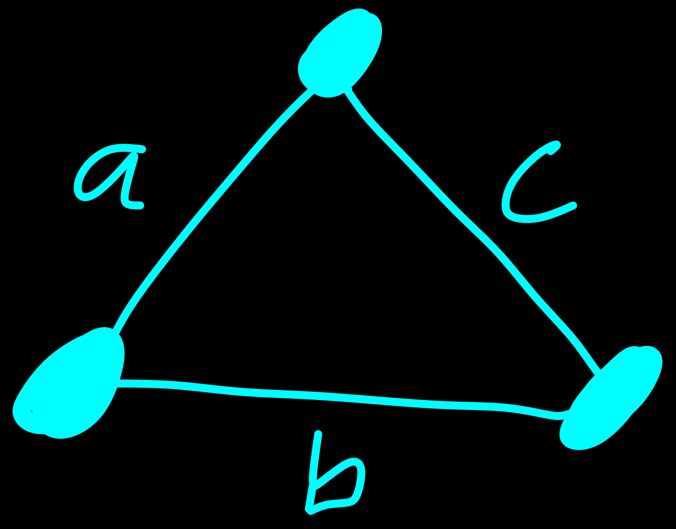


- Question:
  If you know all of the spanning trees in a graph, then do you know the graph itself?
- *What do you mean by "know"?*
- *For a connected graph, If you know:*
  - the edge set of each spanning tree,
  - and any loops in the graph,
  do you know the graph?

An unsatisfactory answer

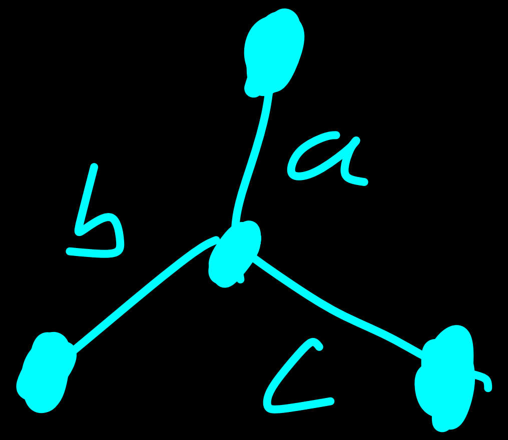
- *If you know:*
  - the edge set of each spanning tree, & any loops in the graph,
  do you know the graph?

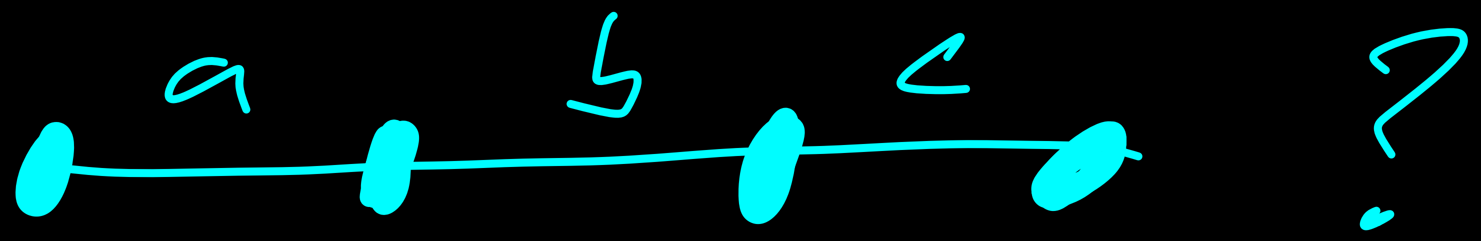
e.g. {a,b}, {b,c}, {a,c} ⇝



- Answer: Clearly a no

e.g. {a,b,c} ⇝

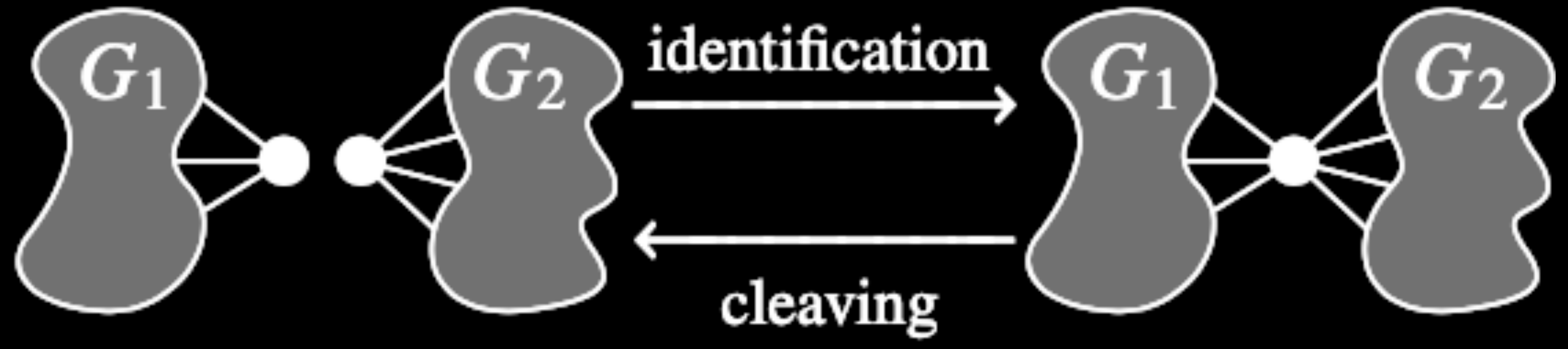




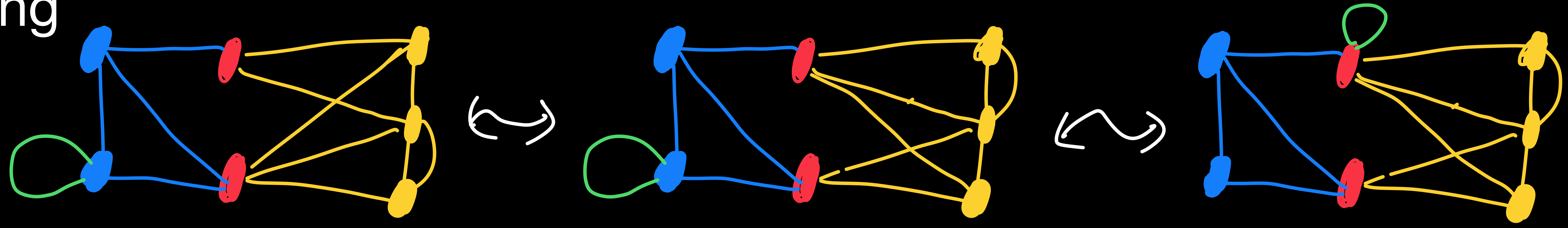
- *But this "no" is really a "yes"…*

Whitney's 2-Isomorphism Theorem

- Two moves:

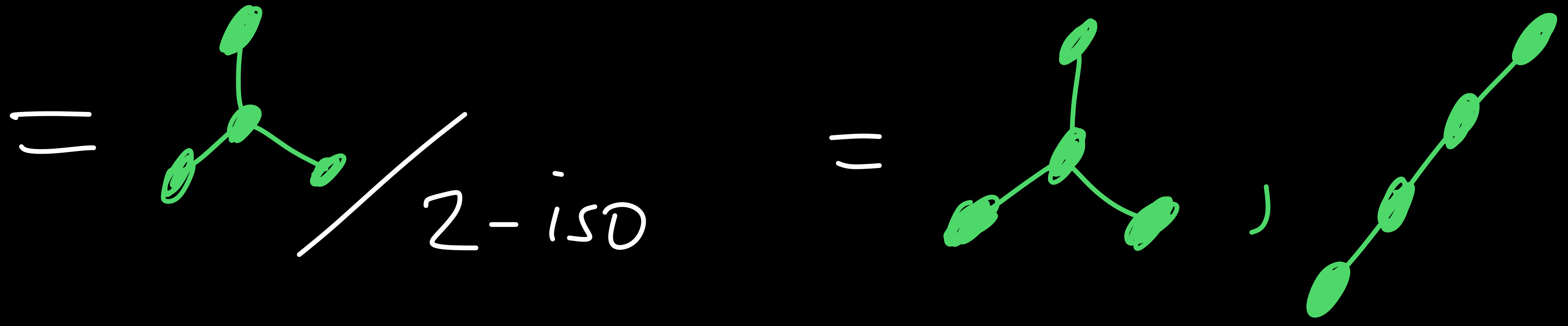


- 2-isomorphism = isomorphism + vertex identification / cleaving + Whitney twisting



- Whitney's 2-Isomorphism Theorem:
  edge set of spanning trees & any loops = graph up to 2-isomorphism

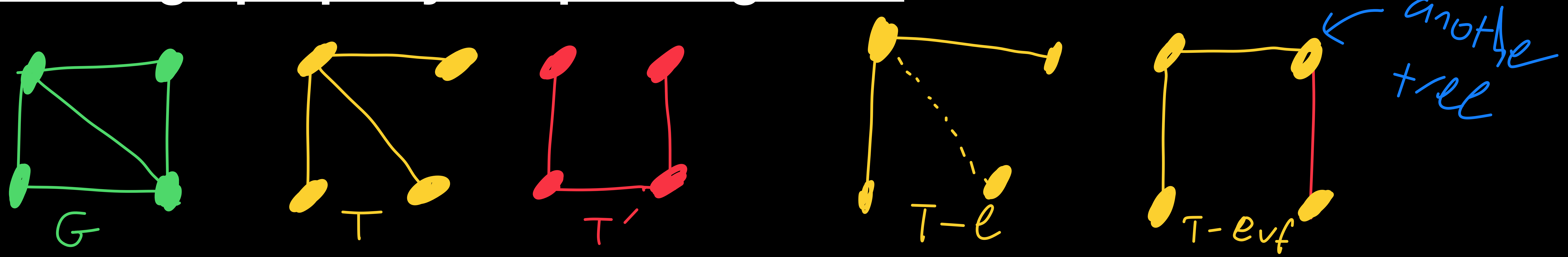
{a,b,c}



- Corollary:
  3-connected graph = edge set of its spanning trees

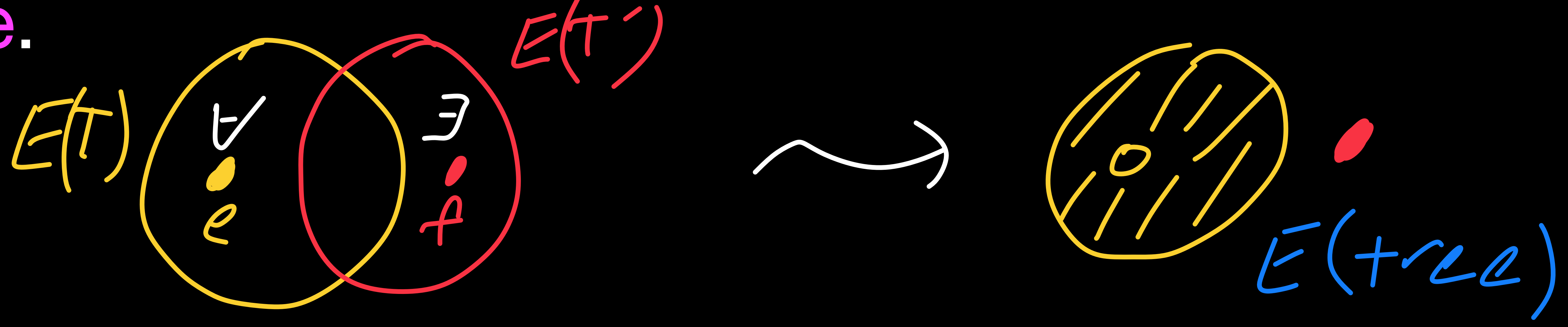
2. The structure of the set of spanning trees

Exchange property of spanning trees



- if T and T' are spanning trees and e is an edge in T but not T', then there is always some edge f in T' but not T such that removing e from T then adding f results in another spanning tree

- But we're not interested in the trees, but the collection of edge sets they give.



- a collection $\mathscr{B}$ of subsets s.t.
  $(\forall A, B \in \mathscr{B})$ $(\forall\, a \in A \setminus B)$ $(\exists\, b \in B \setminus A)$ s.t $(A \setminus a) \cup b \in \mathscr{B}$.

Cycle matroids

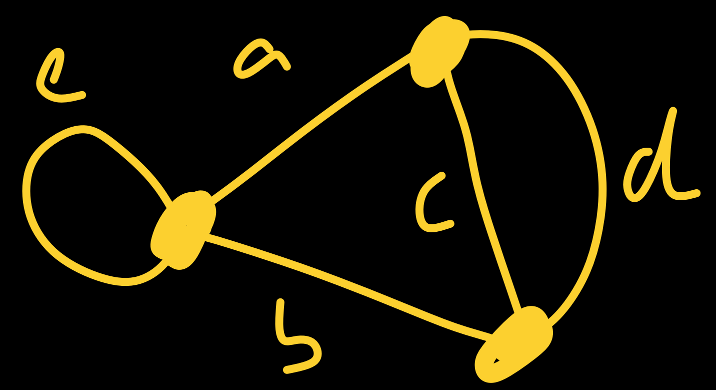
- E be a finite set, $\mathscr{B}$ be a non-empty collection of its subsets

  $(\forall A, B \in \mathscr{B})\ \ (\forall\ a \in A \setminus B)\ (\exists\ b \in B \setminus A)$ s.t $(A \setminus a) \cup b \in \mathscr{B}$.

- The pair $M := (E, \mathscr{B})$ is called a matroid

- Cycle matroid $C(G) := (E, \mathscr{B})$

  E=edge set, $\mathscr{B}$ = { spanning trees }



$E = \{a, b, c, d, e\}$

$\mathscr{B} = \{\{ab\}, \{ac\}, \{ad\}, \{bc\}, \{bd\}\}$

- Whitney's 2-Isomorphism Theorem:

  G and H connected graphs. Then

  $C(G) \cong C(H) \iff G$ and H are 2-isomorphic.

- You can more-or-less work with matroids in place of graphs.

3. The appearance of topology

Algebraic duals

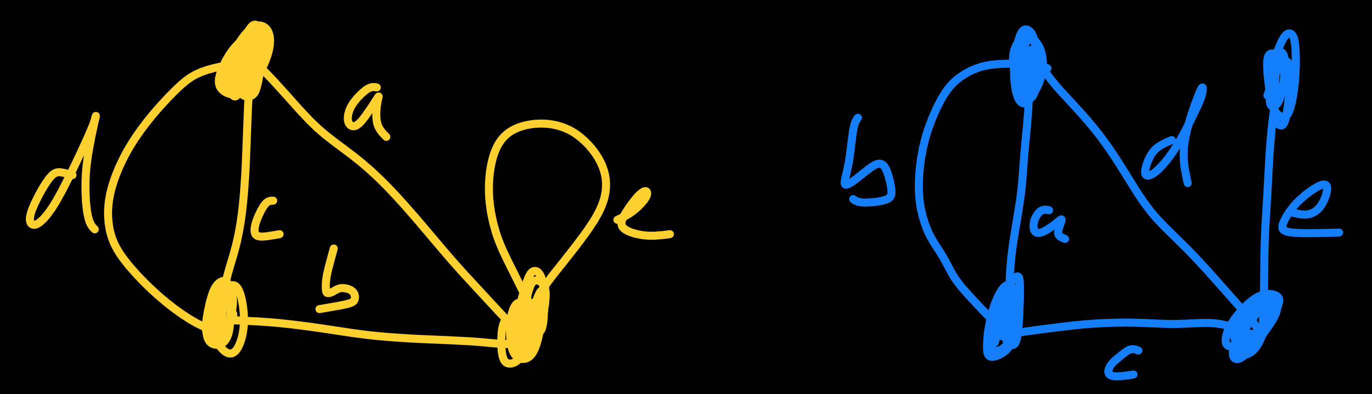
- $M = (E, \mathscr{B})$ a matroid. Its dual is $M^* := (E, \{ E \setminus B : B \in \mathscr{B}\})$

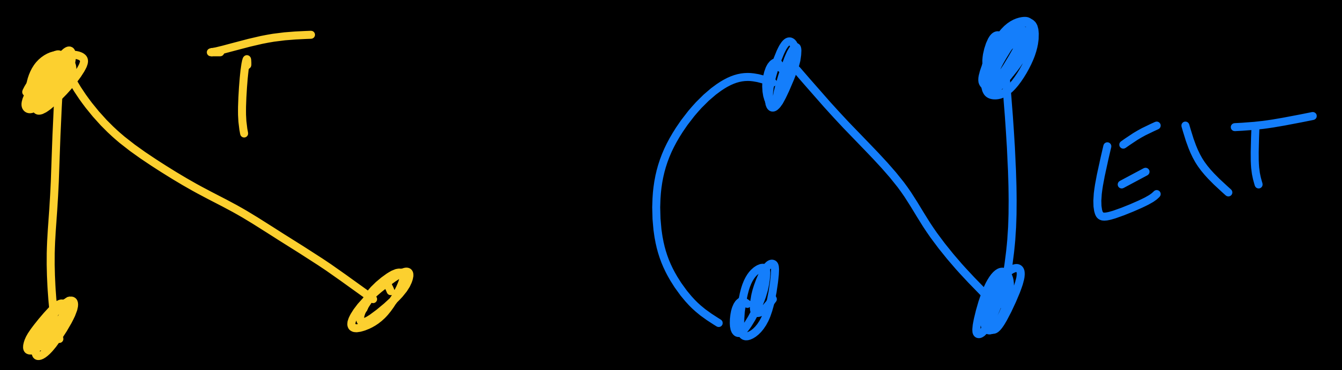
  M= ({abcde} , { {ab}, {ac}, {ad}, {bc}, {bd} })
  M*= ({abcde} , { {cde},{bde},{bce},{ade},{ace} })

- If G is a graph and $C(G)$ its cycle matroid, then the dual matroid $C(G)^*$ is always a matroid. However, it is not always the cycle matroid of a graph. (E.g. C(K_5) does not come from a graph.)

- When does $C(G)^*$ come from a graph?

- Graphs G and H are algebraic duals if T a spanning tree of G $\iff$ $E \setminus T$ a spanning tree of H



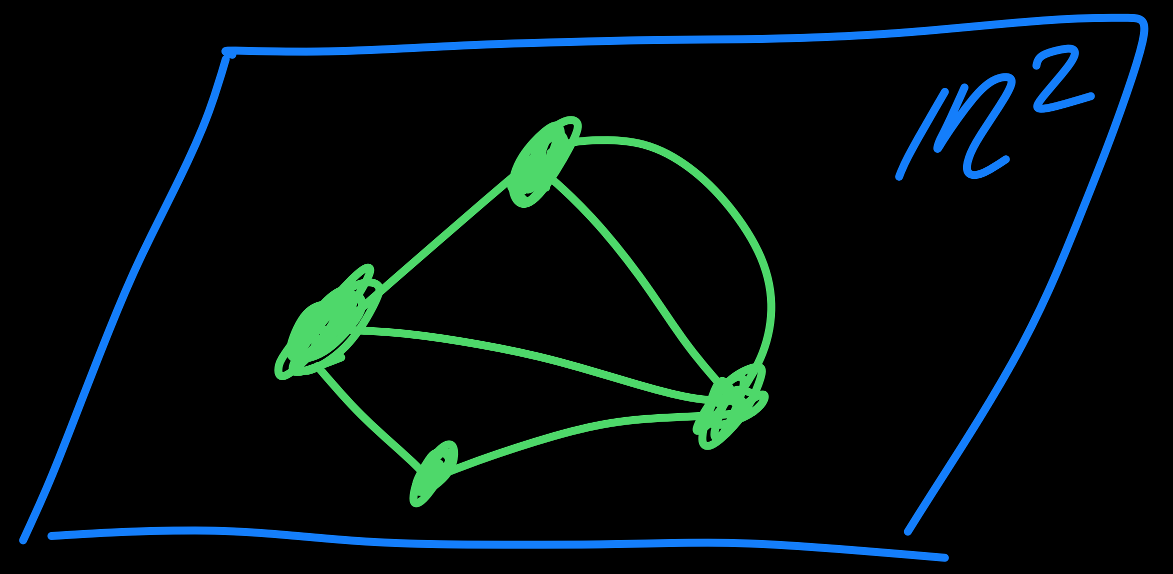


- $C(G)^*$ comes from a graph $\iff$ G has an algebraic dual

- May or may not exist. May or may not be unique.

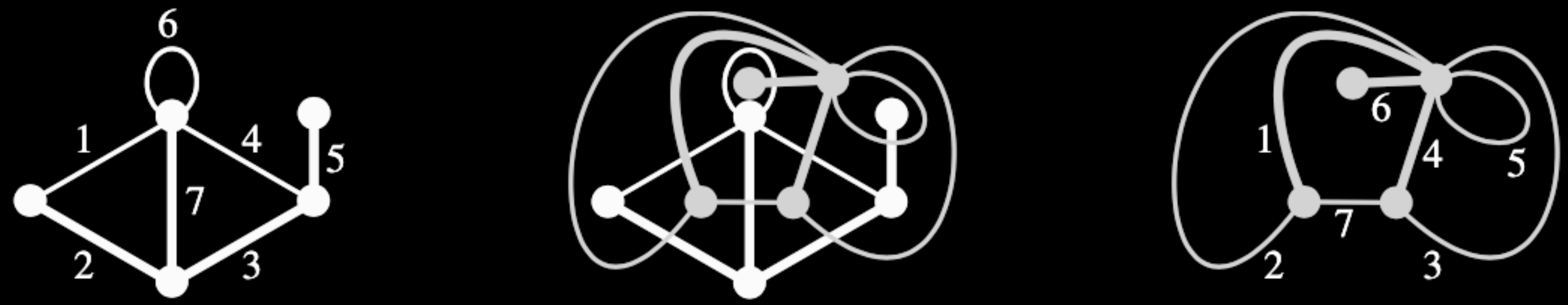
Geometric duals

- The existence of algebraic duals is tied to the topological properties of a graph.
- plane graph = a connected graph drawn plane / sphere
- planar = can be drawn in the plane / sphere



- Geometric dual $\mathbb{G}^*$ of plane graph $\mathbb{G}$

  vertices of $\mathbb{G}^*$ = faces of $\mathbb{G}$, edge of $\mathbb{G}^*$ when faces of $\mathbb{G}$ adjacent



- Geometric duals are always algebraic duals
- Algebraic duals are always geometric duals
- Collecting this together….

Whitney's Theorems in terms of matroids:

- G connected
  Cycle matroid: C(G) = (E, {spanning trees})

- Then the dual matroid $C(G)^*$ is the cycle matroid of a graph $\Longleftrightarrow$ if G is planar

- if G is planar then

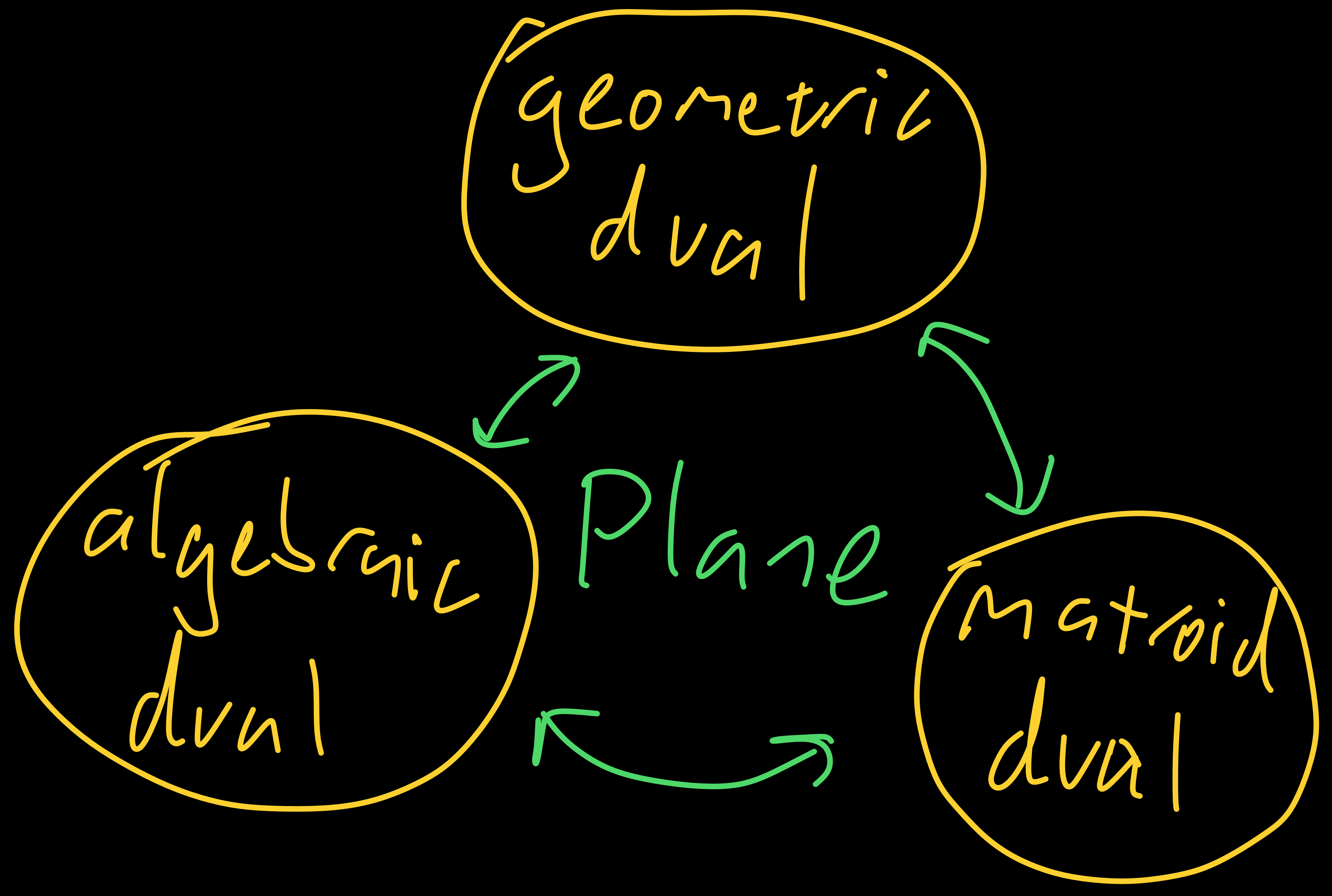
  $$C(G)^* = C(G^*),$$

  where G^* is the geometric dual of *any* plane embedding of G.

- $C(G)$ and algebraic duals unique up to 2-isomorphism.

It’s all about the plane

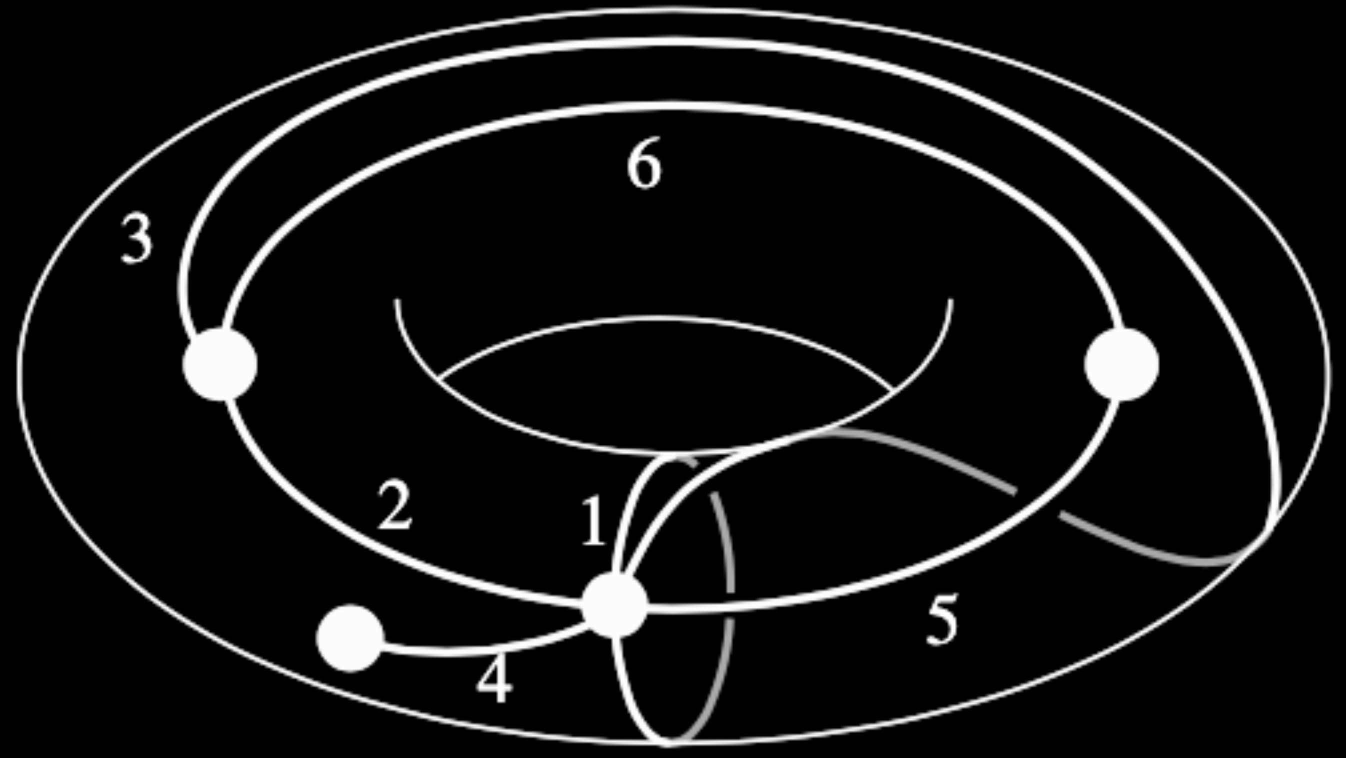
- We have seen:
- Spanning tree structure ↔ topological structure
- But tied to planarity



- What if you do not want to restrict yourself to plane or planar graphs?

4. Moving away from the plane

Surfaces and embedded graphs

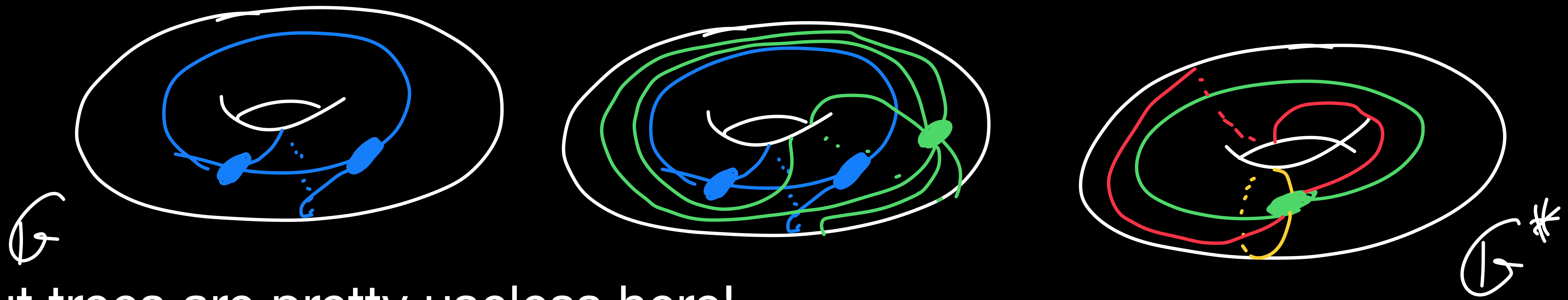


- Orientable surfaces (for simplicity)

- Embedded graph = drawn on surface + edges don't cross + faces are discs

- Geometric dual $\mathbb{G}^*$ of $\mathbb{G}$: as before

  vertices of $\mathbb{G}^*$ = faces of $\mathbb{G}$ & edge of $\mathbb{G}^*$ when faces of $\mathbb{G}$ adjacent

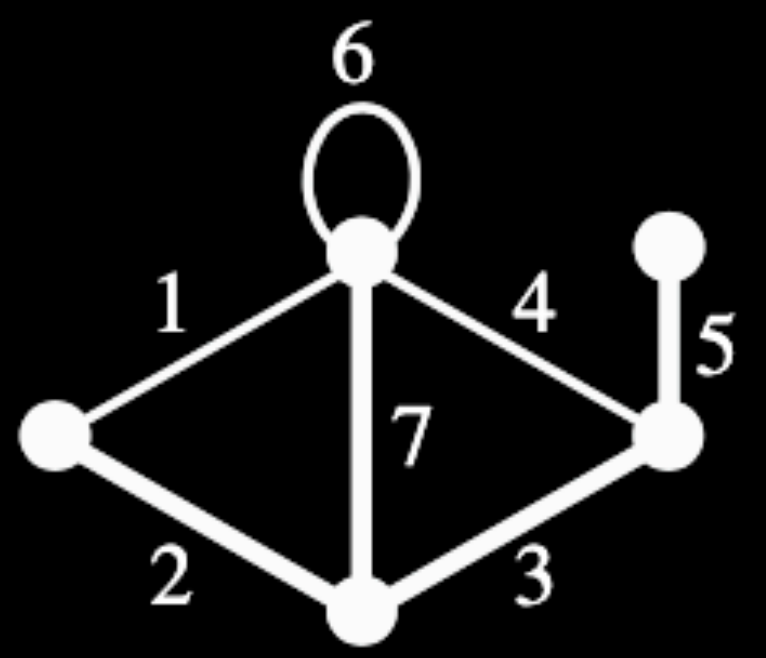


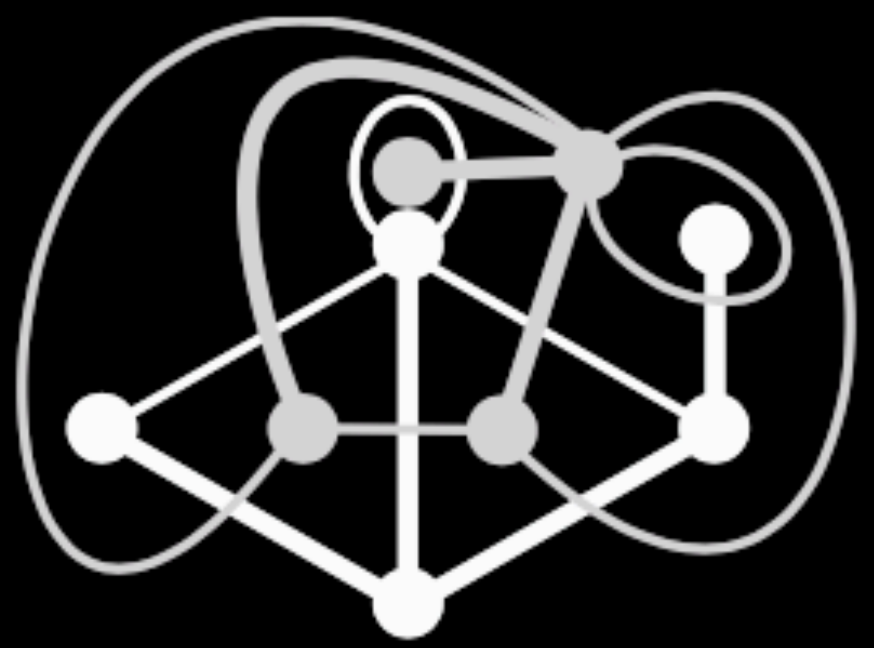
- But trees are pretty useless here!

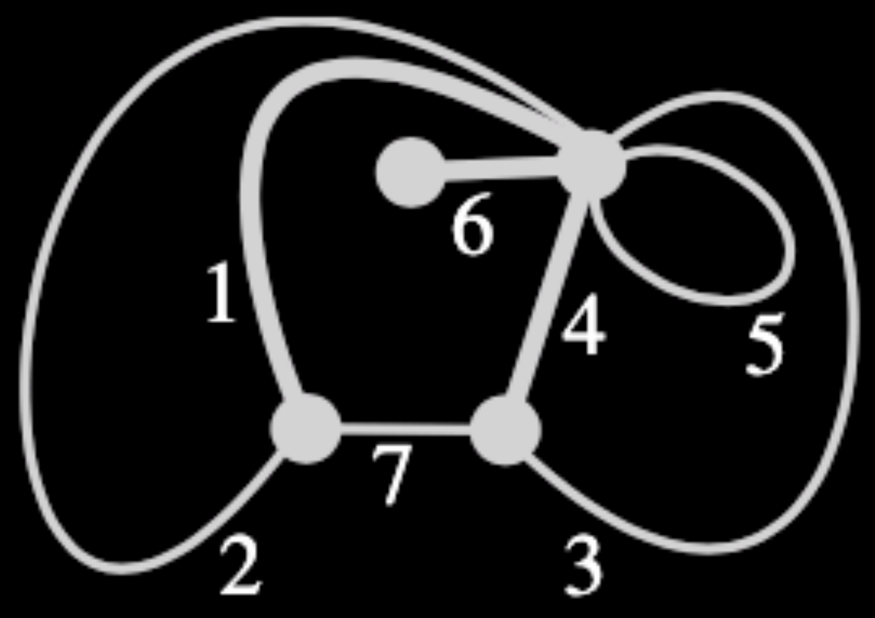
What is a "tree" for an embedded graph?

- Plane graphs: (geometric = algebraic = matroid)

- T a tree in $\mathbb{G} \iff E \setminus T$ a tree in $\mathbb{G}^*$

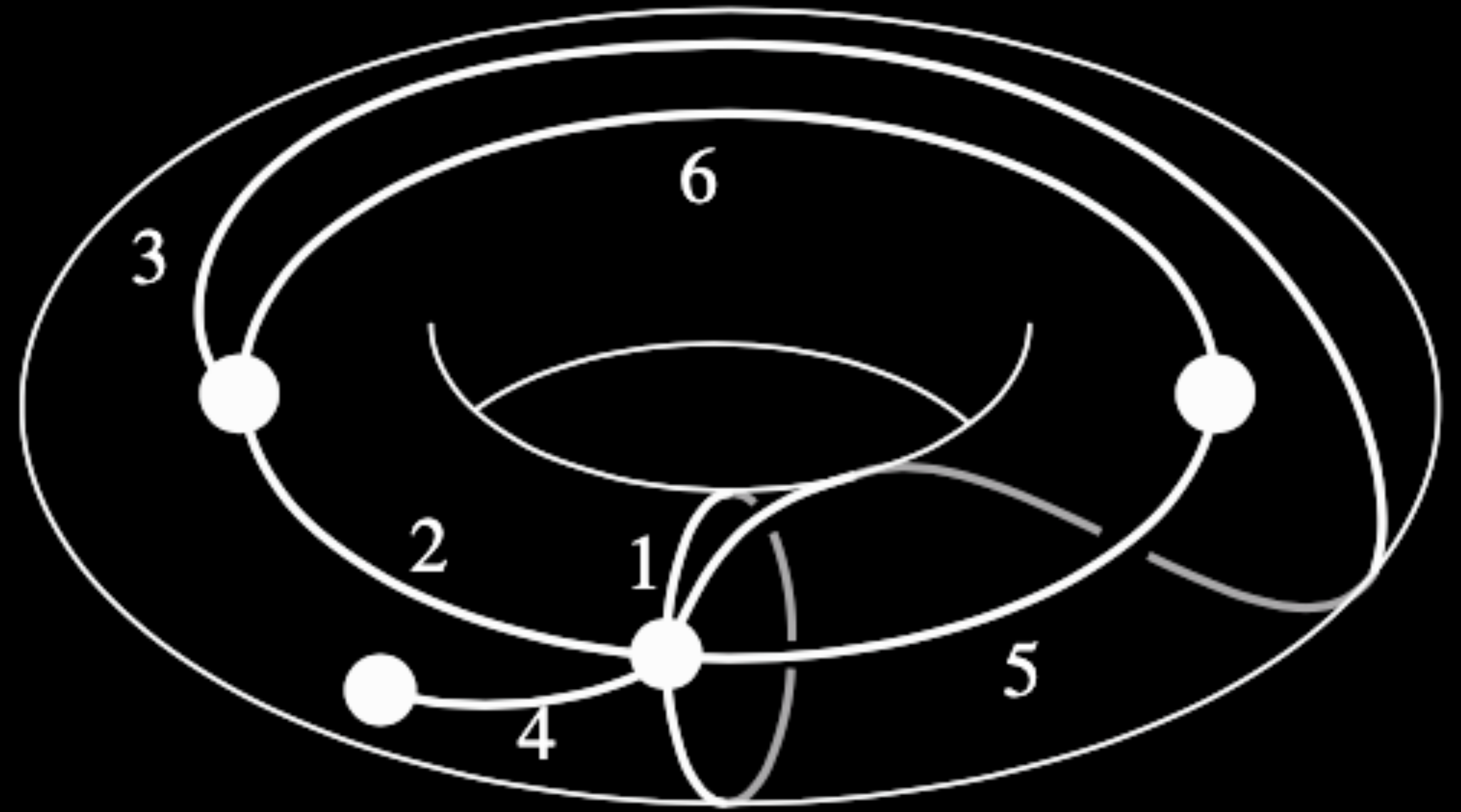






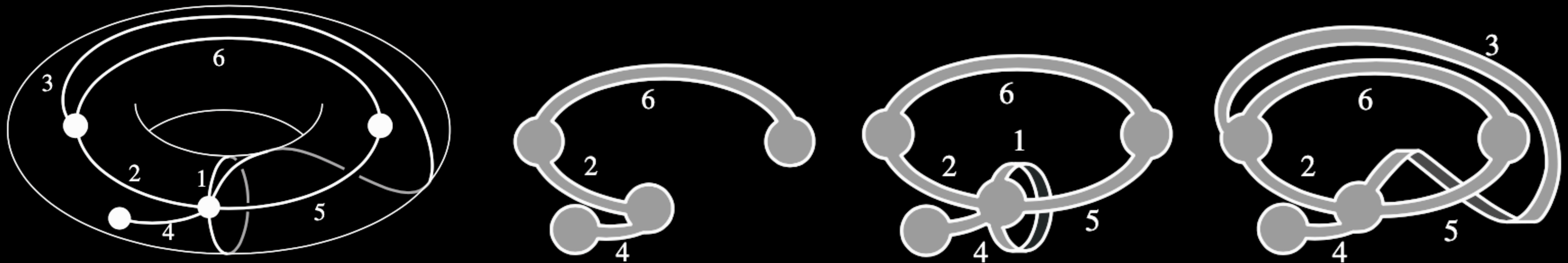
- Embedded graphs:

  T a tree in $\mathbb{G} \iff E \setminus T$ has one-face in $\mathbb{G}^*$



What is a “tree” for an embedded graph?

- Spanning quasi-tree = neighbourhood has exactly one boundary component.



- plane ⇝ surface

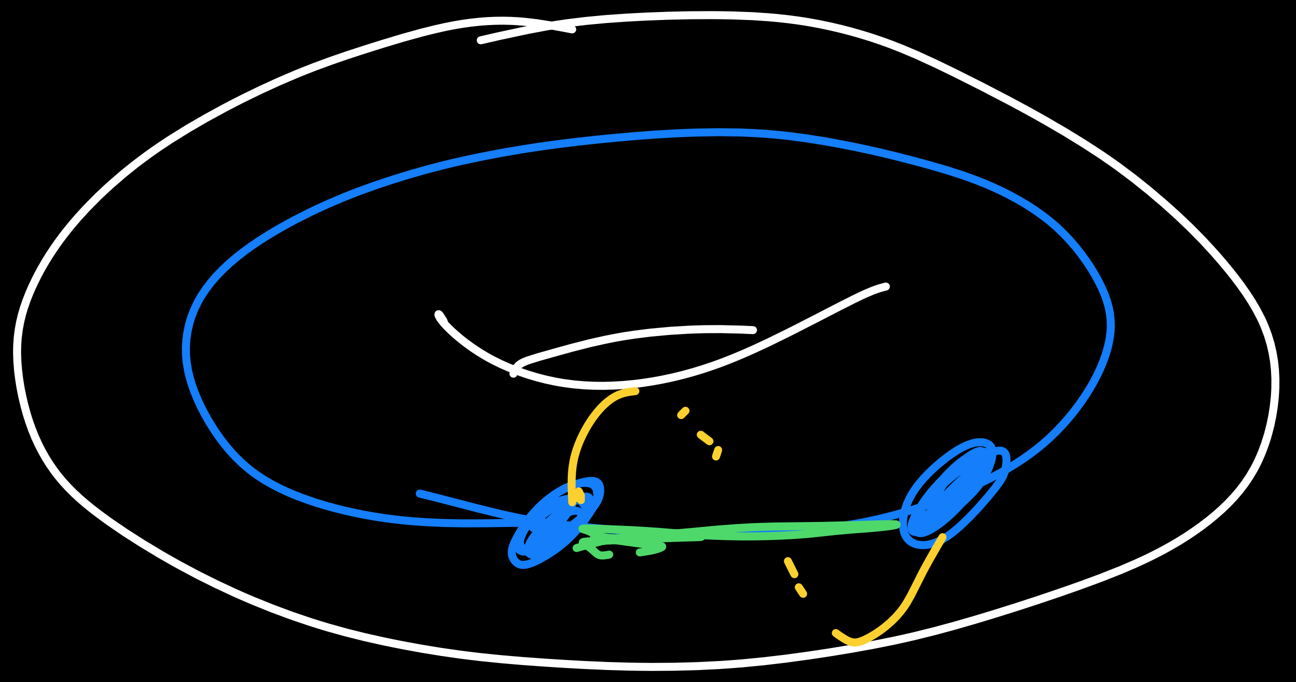
  Trees ⇝ quasi-trees

A new question

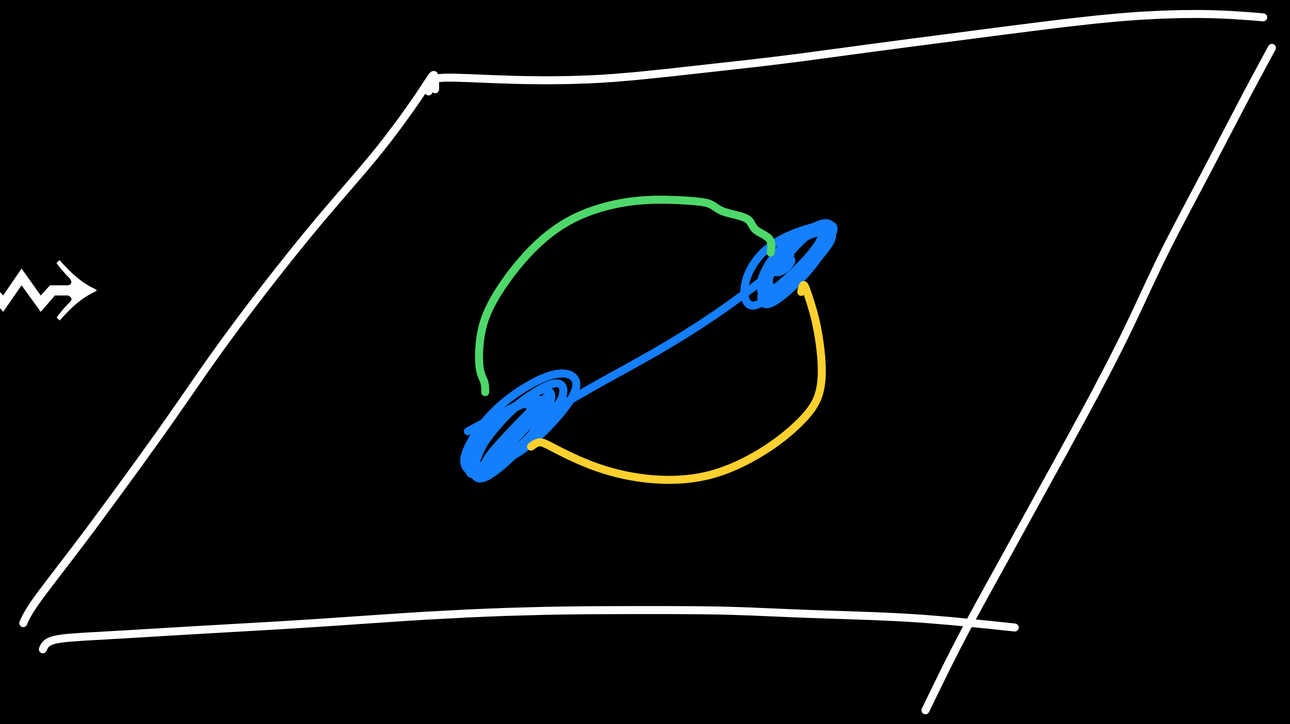
- Previously:
  If you know all of the spanning trees in a graph, then do you know the graph itself?

- Topogical version:
  If you know the edge sets of all of the spanning quasi-trees in an embedded graph, then do you know the embedded graph itself?

{abc},{a},{b},{c} ⇝



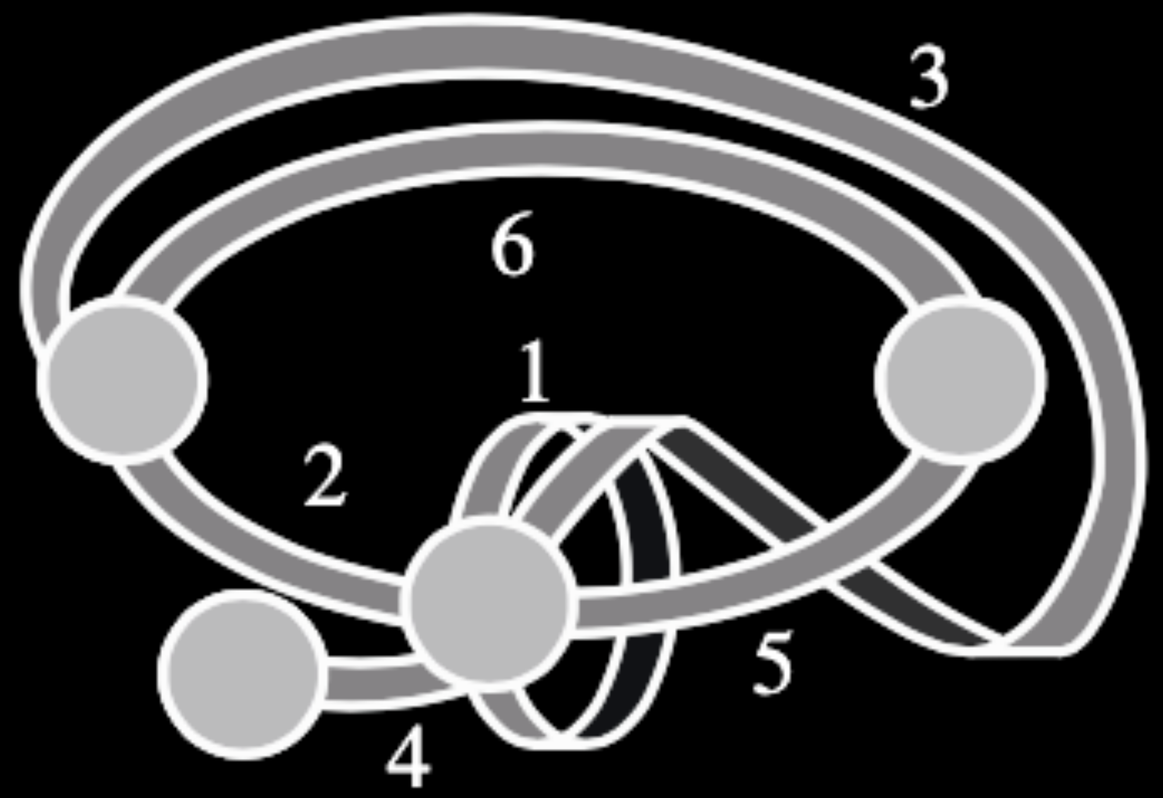
{a},{b},{c} ⇝

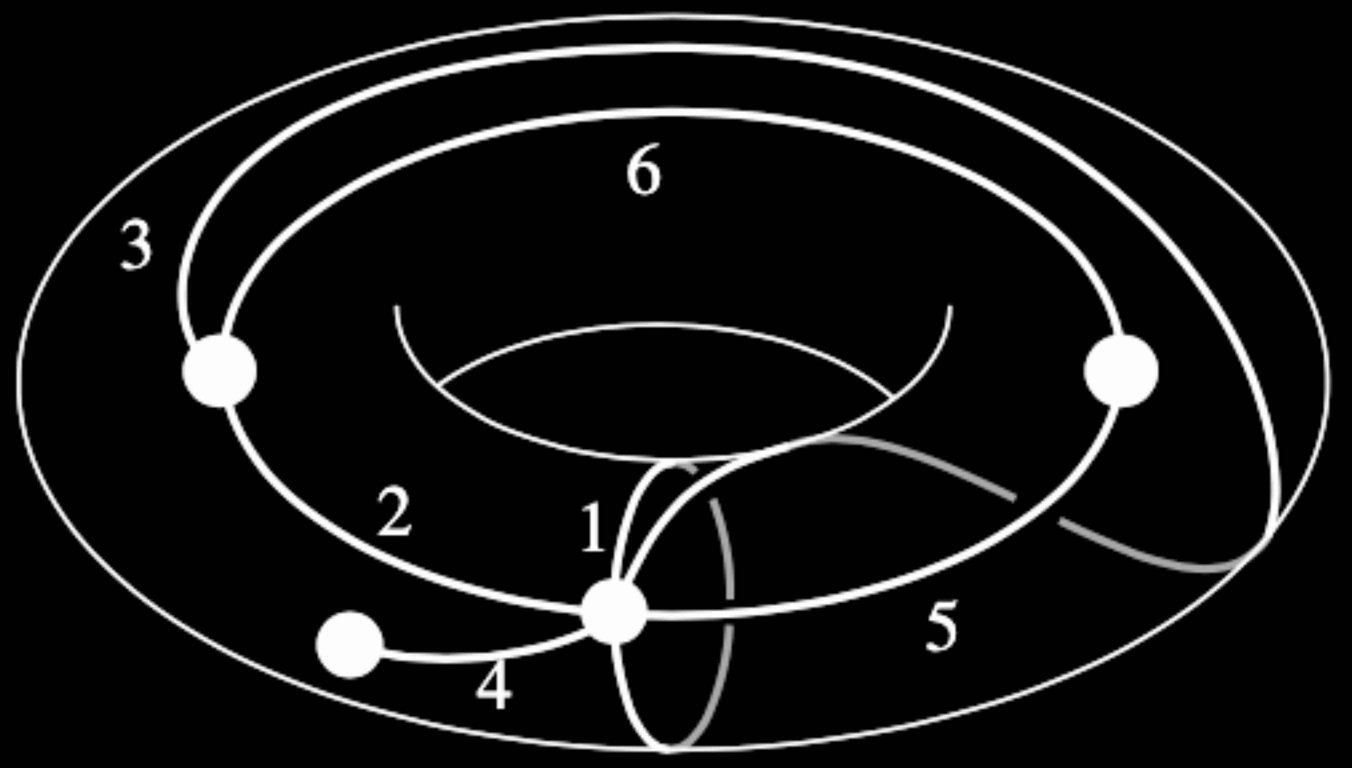


4. Duals and partial duals

From embedded graphs to Ribbon graphs

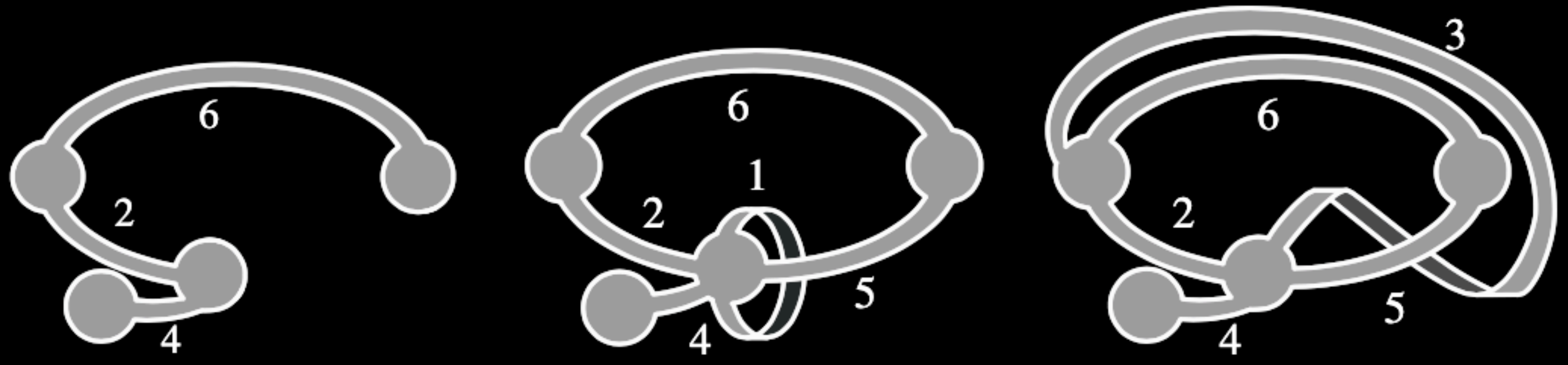
- Ribbon graph = "graphs whose vertices consist of discs, and whose edges consist of ribbons"





- Ribbon graphs = embedded graphs

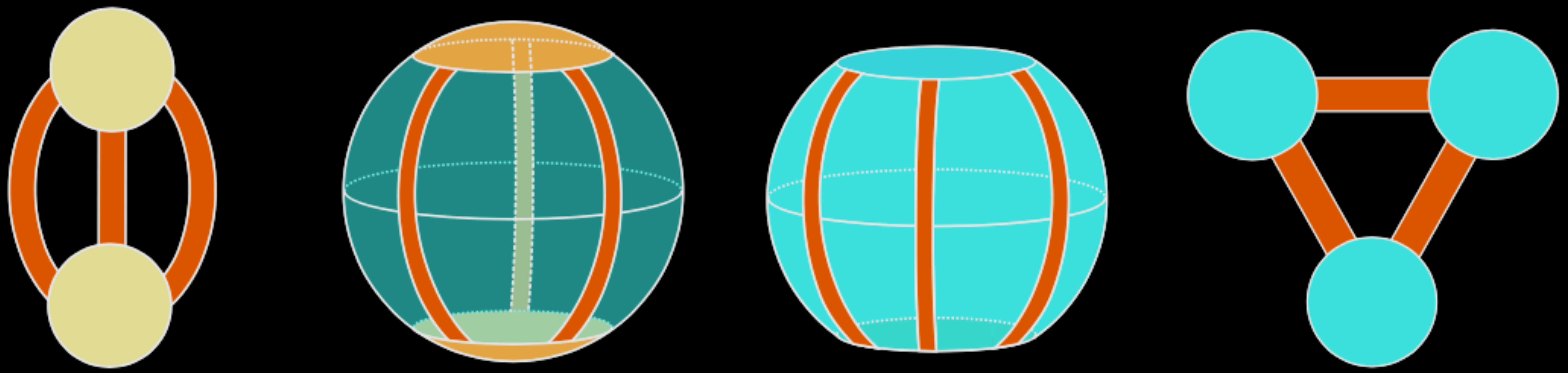
- Spanning quasi-tree = all vertices & one boundary component.



Duality revisited

- Slick way to construct dual $\mathbb{G}^*$ of $\mathbb{G}$:

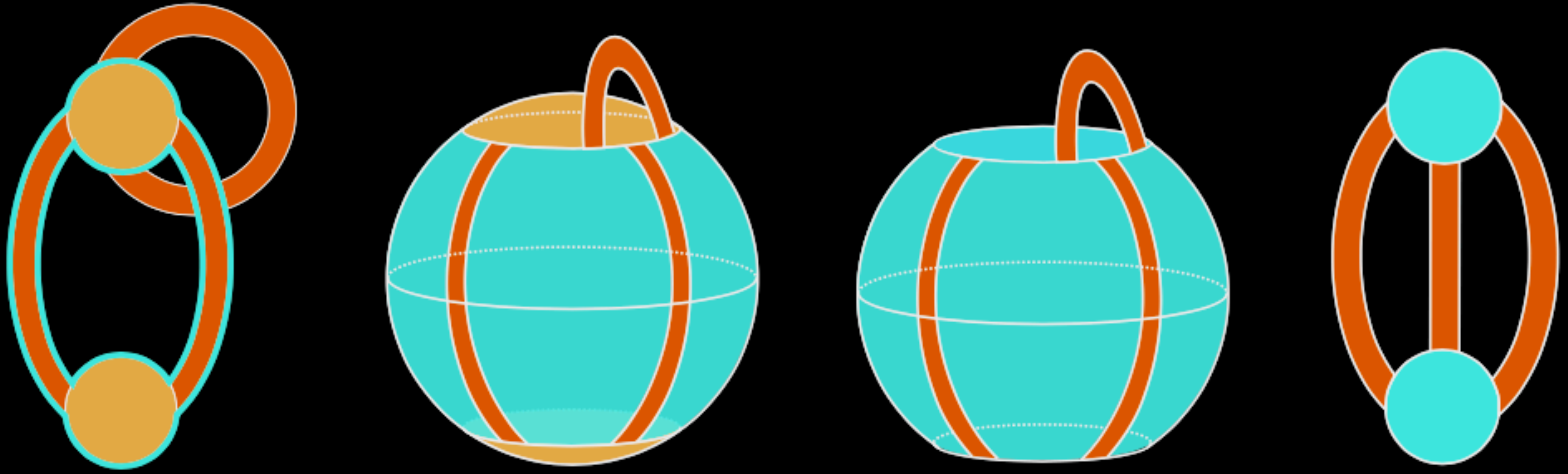
  glue a disc to each boundary component of $\mathbb{G}$, then delete old vertices



- New idea: [Chmutov '09] dual only some of the edges by gluing discs to the boundary of a subgraph

Partial duality

- New idea [Chmutov '09]: dual only some of the edges by gluing discs to the boundary of a subgraph



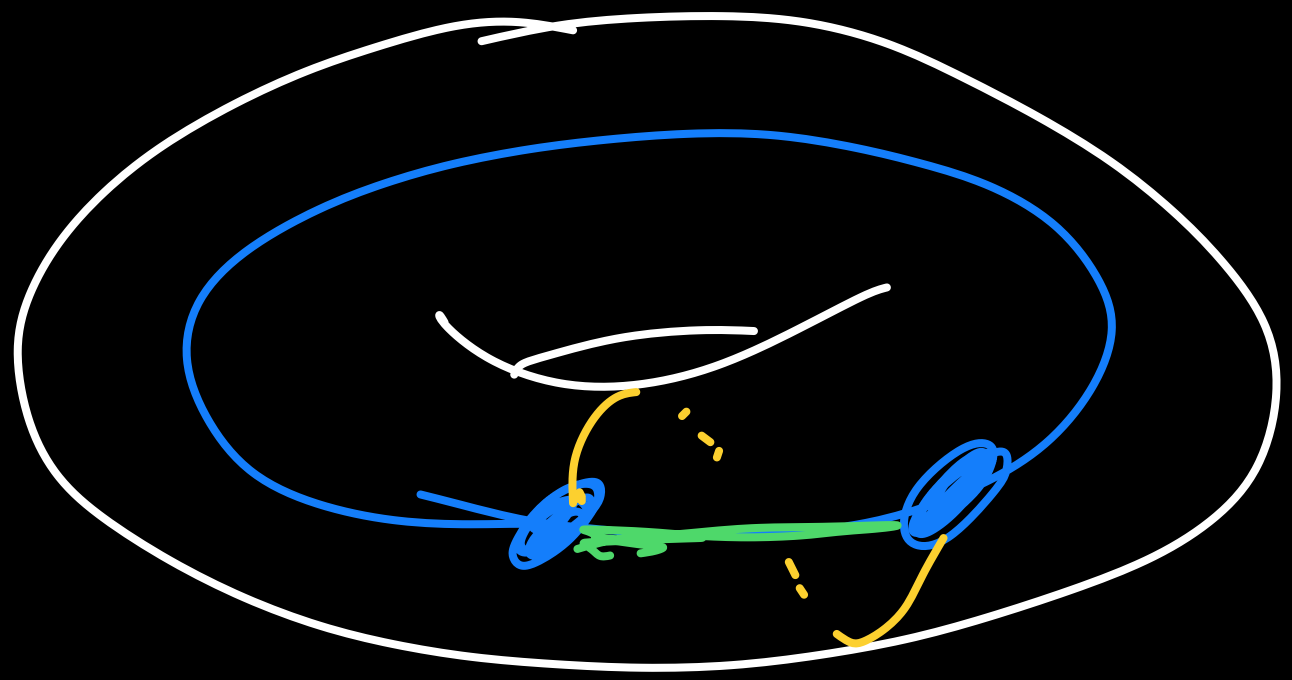
- Partial Dual $\mathbb{G}^A$: dual of $\mathbb{G}=(V,E)$ w.r.t. set of edges A by glue discs each boundary component of (V,A) in $\mathbb{G}$, then delete old vertices

- Forms geometric dual $\mathbb{G}^*$ one edge at a time!

5. The structure of the set of quasi-trees

A cycle matroid for ribbon graphs [Chun — M. — Noble — Rueckriemen ’09]

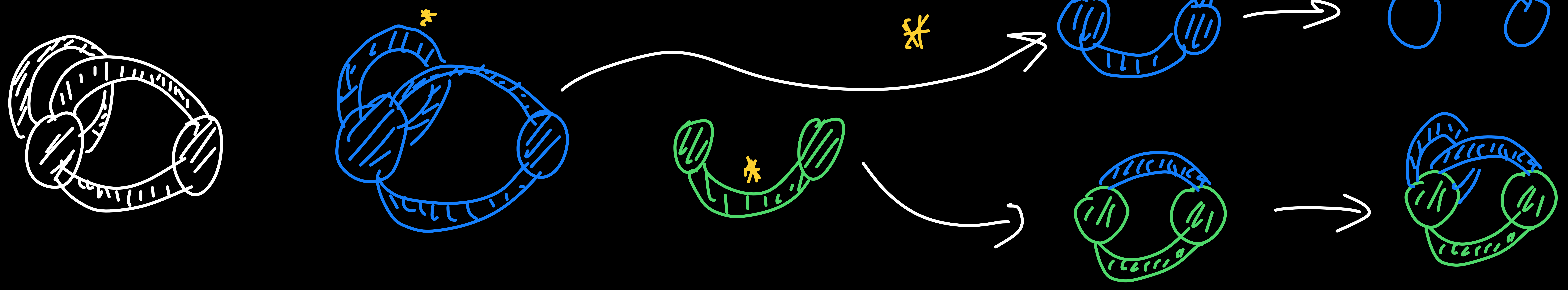
- Recall for graphs: spanning trees $\rightsquigarrow$ exchange properties $\rightsquigarrow$ matroids
- Let’s mirror this construction for ribbon graphs:
  spanning trees for graphs $\rightsquigarrow$ spanning quasi-trees for ribbon graphs
- Cycle matroid of graph: C(G):=(E, {spanning trees})
  $\rightsquigarrow$
  delta-matroid of ribbon graph: D($\mathbb{G}$):=(E, {spanning quasi-trees})



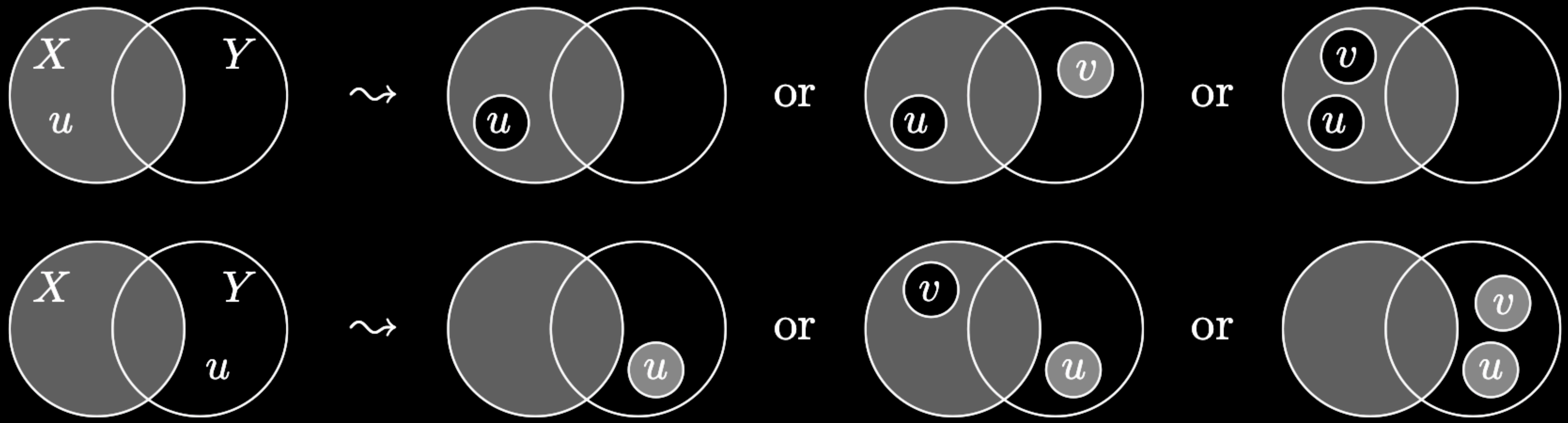
D($\mathbb{G}$)=({abc} , { {abc},{a},{b},{c} })

The exchange property

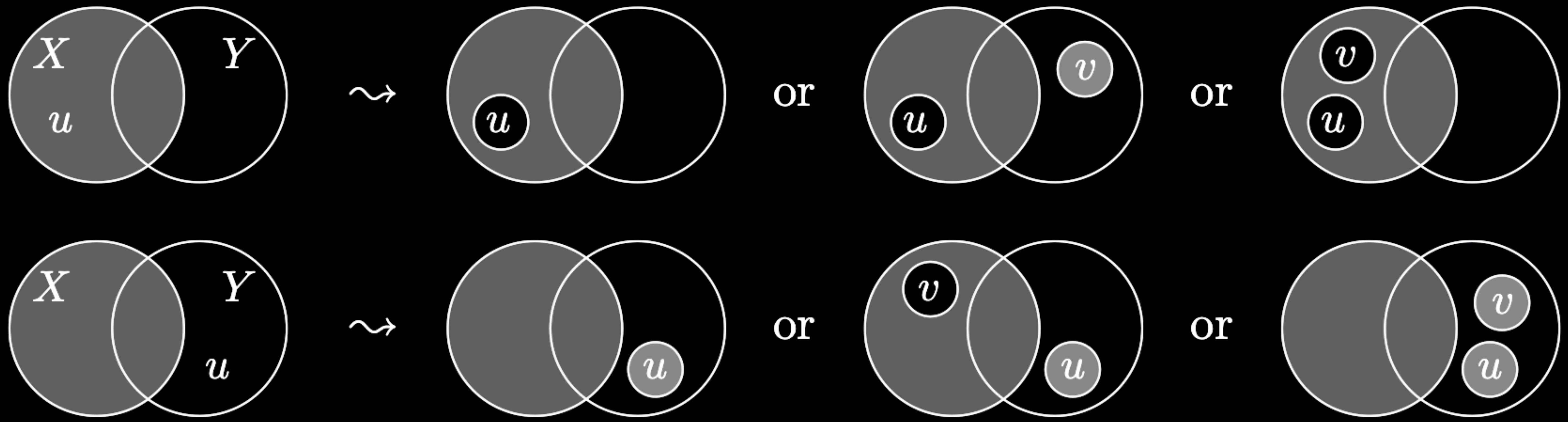
- You can similarly move between quasi-trees, but you have to be able to add and remove edges:



- $(\forall X,Y\in\mathcal{Q})\ (\forall u\in X\Delta Y)\ (\exists v\in X\Delta Y)\ (X\Delta\{u,v\}\in\mathcal{Q})$.



The exchange property



- [Bouchet 80's] A delta-matroid $D = (E, \mathscr{F})$ where:

  E is a finite set,

  $\mathscr{F}$ a non-empty collection of its subsets.

  $\mathscr{F}$ satisfies the Symmetric Exchange Axiom:

  $(\forall X, Y \in \mathscr{F})\ (\forall u \in X \Delta Y)\ (\exists v \in X \Delta Y)\ (X \Delta \{u, v\} \in \mathscr{F})$.

- D($\mathbb{G}$):=(E, {spanning quasi-trees}) is a delta-matroid.

6. Completing Whitney’s Theorems

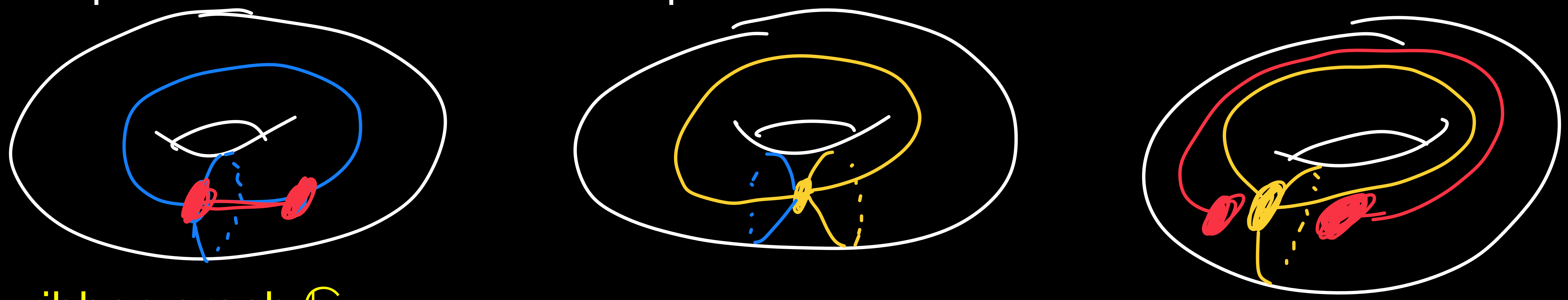
Duality

- $M = (E, \mathscr{B})$ a matroid. Its dual is $M^* := (E, \{ E \setminus B : B \in \mathscr{B}\})$

- D = $(E, \mathscr{F})$ a delta-matroid. Its dual is D* := $(E, \{ E \setminus F : F \in \mathscr{F}\})$

  D($\mathbb{G}$) =({abc} , { {abc}, {a}, {b}, {c} })
  D($\mathbb{G}^*$)=({abc} , { ∅ , {bc}, {ac}, {ab} })

- But Q is a quasi-tree in $\mathbb{G} \iff E \setminus Q$ a quasi-tree in $\mathbb{G}^*$



- For any ribbon graph $\mathbb{G}$

  D($\mathbb{G}^*$) = D($\mathbb{G}$)*

- But wait, there is more: Q is a quasi-tree in $\mathbb{G} \iff A \triangle Q$ a quasi-tree in $\mathbb{G}^A$

Partial duality for delta-matroids

- Recall: D = $(E,\mathscr{F})$ a delta-matroid. Its dual is

  D* := $(E$, { E \ F : F ∈ $\mathscr{F}$}) =$(E$, {in exactly one of E or F : F ∈ $\mathscr{F}$}) = $(E$, {E △ F : F ∈ $\mathscr{F}$})

  D($\mathbb{G}$) =({abc} , { {abc}, {a}, {b}, {c} }   D($\mathbb{G}$*)=({abc} , { ∅ , {bc}, {ac}, {ab} })

- D = $(E,\mathscr{F})$ a delta-matroid. Its partial dual is

  D^A := $(E$, {in exactly one of A or F : F ∈ $\mathscr{F}$}) = $(E$, {A △ F : F ∈ $\mathscr{F}$})

  D($\mathbb{G}$) =({abc} , { {abc}, {a}, {b}, {c} })

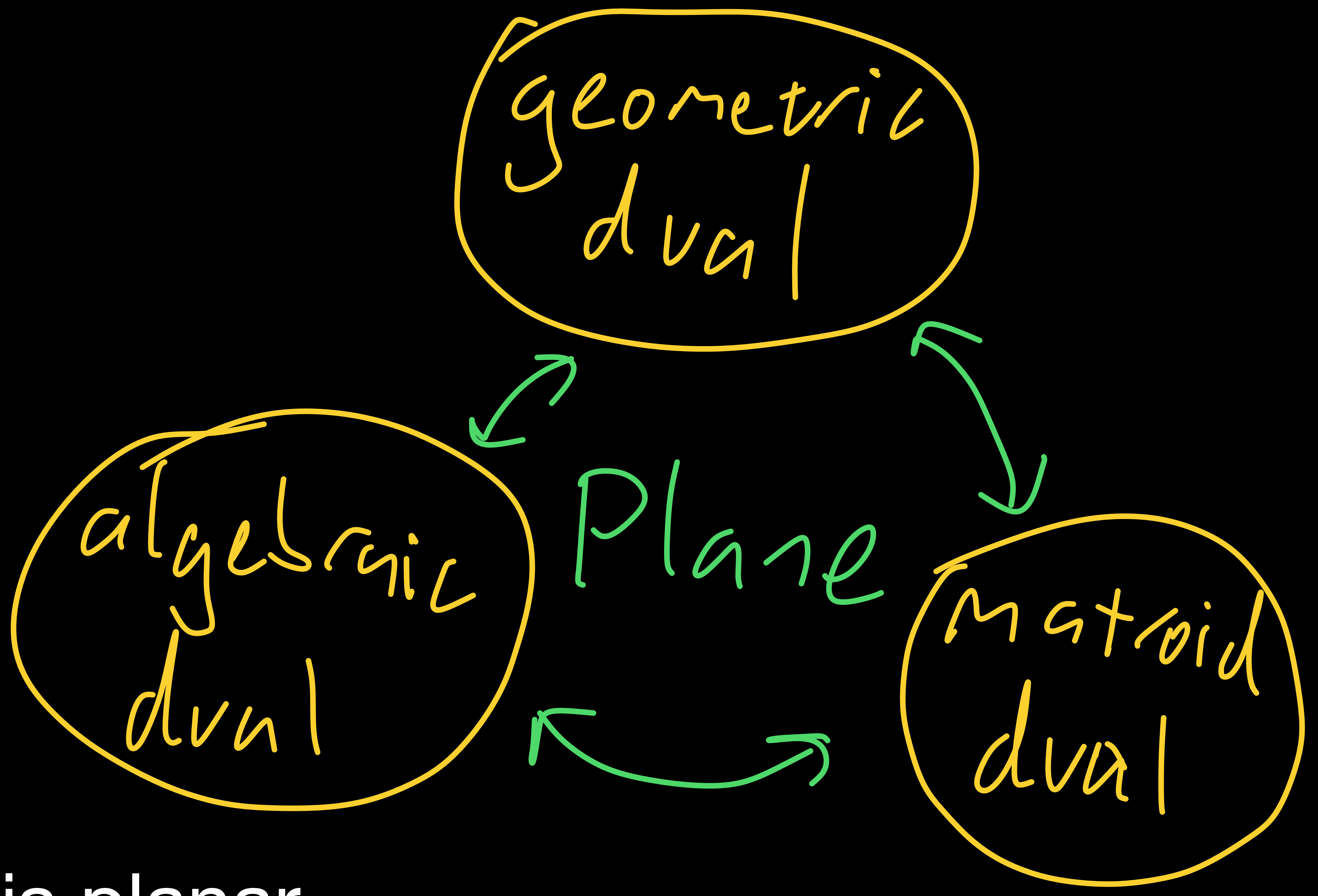
  D($\mathbb{G}^{\{ab\}}$)=({abc} , { {c}, {b}, {a}, {abc} })

- But Q is a quasi-tree in $\mathbb{G}$ $\iff$ Q △ A a quasi-tree in $\mathbb{G}^A$

- [Chun—M.—Noble—Rueckriemen ’09] For any ribbon graph $\mathbb{G}$

  $D(\mathbb{G}^A) = D(\mathbb{G})^A$

Completing Whitney's Theorems

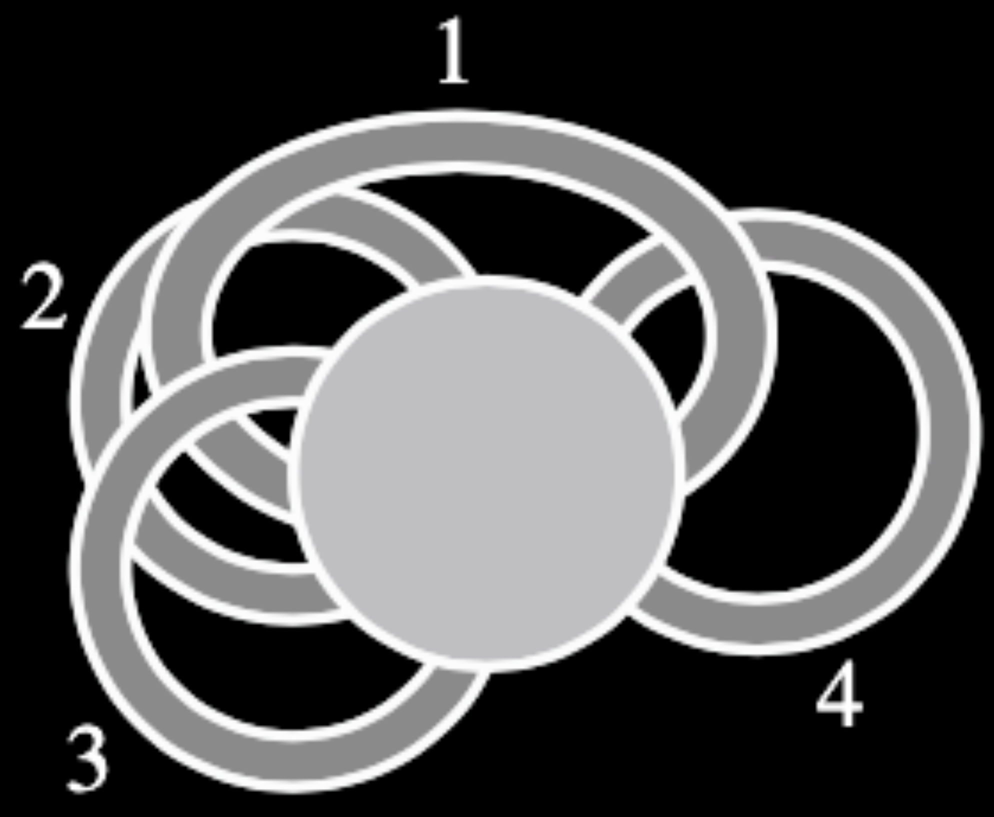


- G graph
  $\mathbb{G}$ ribbon graph
- Cycle matroid: C(G) = (E, {spanning trees})
  delta-matroid: D($\mathbb{G}$) = (E, {spanning quasi-trees})
- $C(G)^*$ is the cycle matroid of a graph if and only if G is planar
  D($\mathbb{G}$)* is *always* the delta-matroid of an ribbon graph
- if G is *planar* then $C(G)^* = C(G^*)$.
  $D(\mathbb{G})^* = D(\mathbb{G}^*)$ for *every* ribbon graph
  $D(\mathbb{G}^A) = D(\mathbb{G})^A$ for *every* ribbon graph
- $C(G)$ and algebraic duals unique up to 2-isomorphism.
  We had better take a look at this one!

7. Do the quasi-trees determine the ribbon graph?

Bouquets

- Bouqet = one-vertex ribbon graph



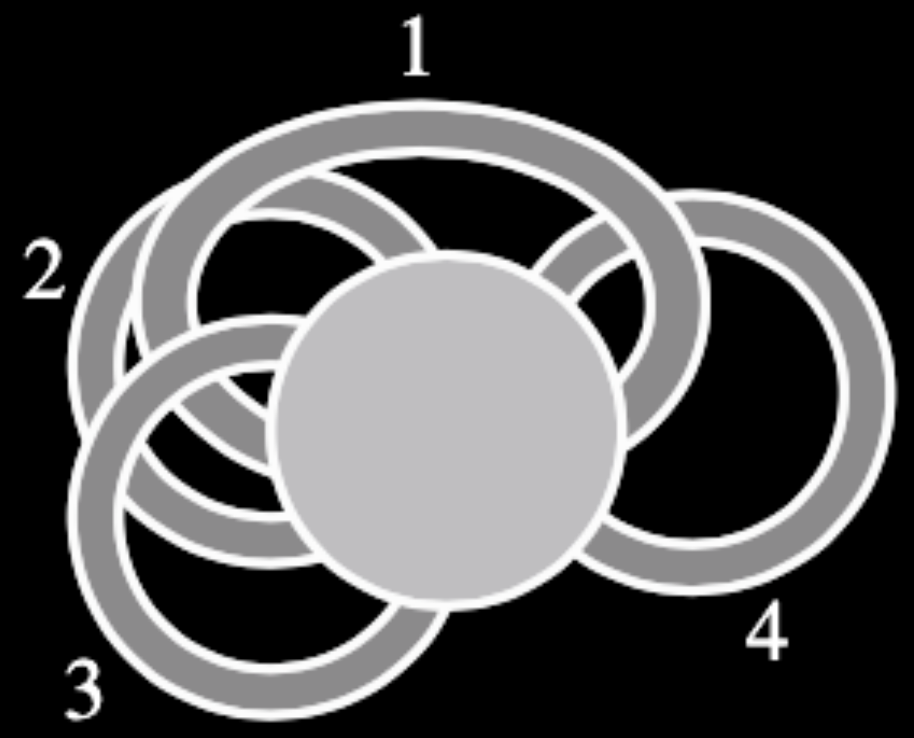
- Every ribbon graph has a partial dual that is a bouquet (e.g., partial dual along a spanning tree)
- Then since $D(\mathbb{G}^A) = D(\mathbb{G})^A$ we can work with bouquets rather than ribbon graphs in general.
- And we can take advantage of a method from algebraic topology for determining via a matrix if an orientable bouquet is a quasi-tree.

Matrices

- construct an $|E| \times |E|$-matrix $\mathbf{IM}_{\mathbb{G}}$ by setting (e,f)-entry to be:

  1 if edges e and f are interlaced (form a genus 1 ribbon subgraph)

  0 otherwise.



$$\mathbf{IM}_{\mathbb{G}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

- over GF(2) we have $\det(\mathbf{IM}_{\mathbb{G}}) = 1 \iff \mathbb{G}$ is a quasi-tree

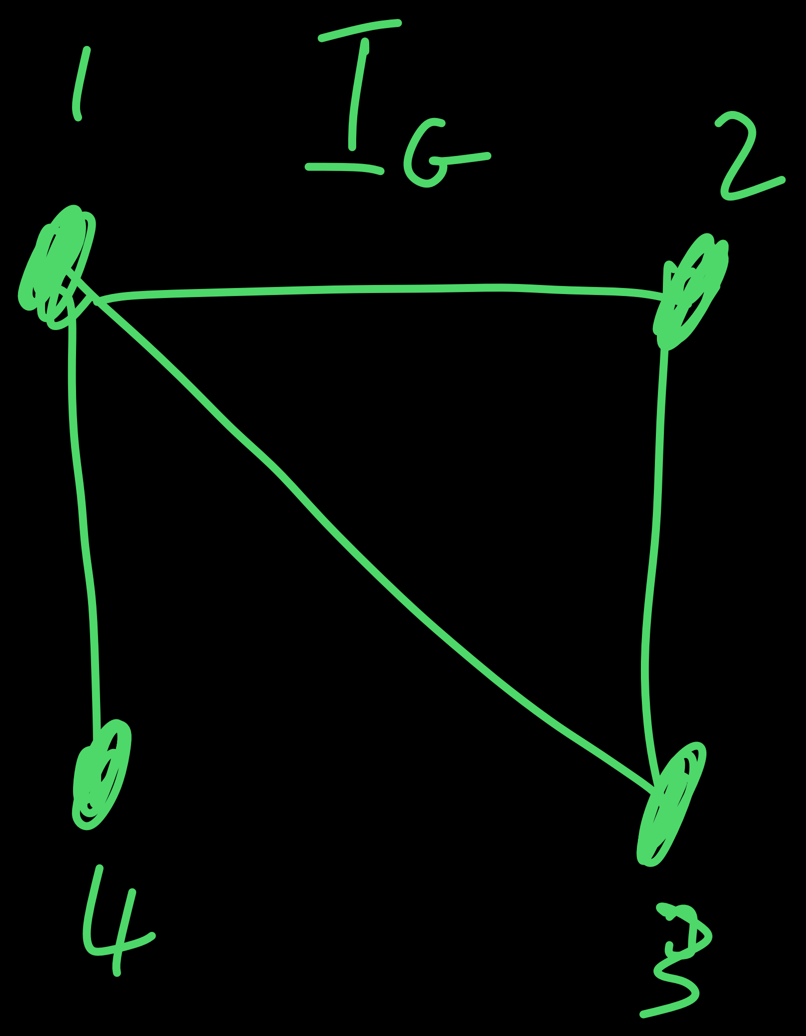
- Thus $D(\mathbb{G})$ is completely determined by $\mathbf{IM}_{\mathbb{G}}$

  (X edge set of quasi-tree $\iff$ principal submatrix $\mathbf{IM}_{\mathbb{G}}[X]$ is non-singular)

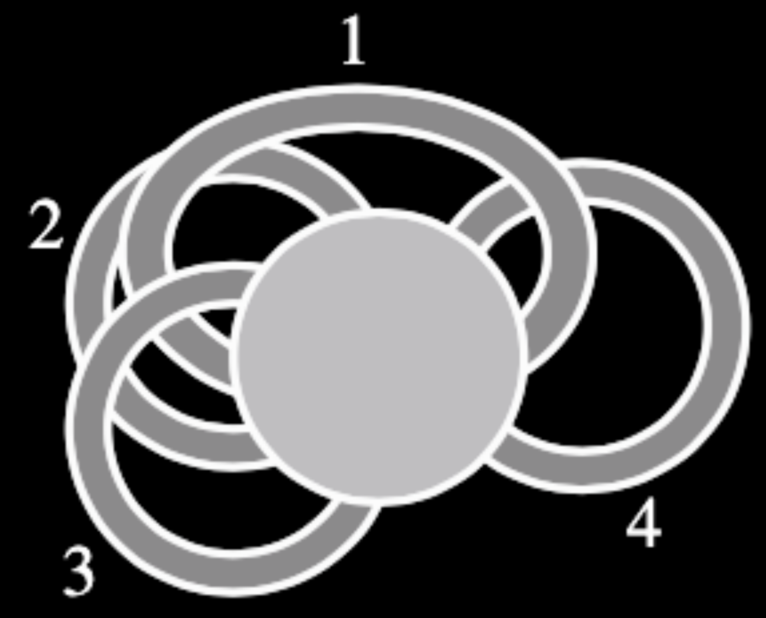
Intersection graphs

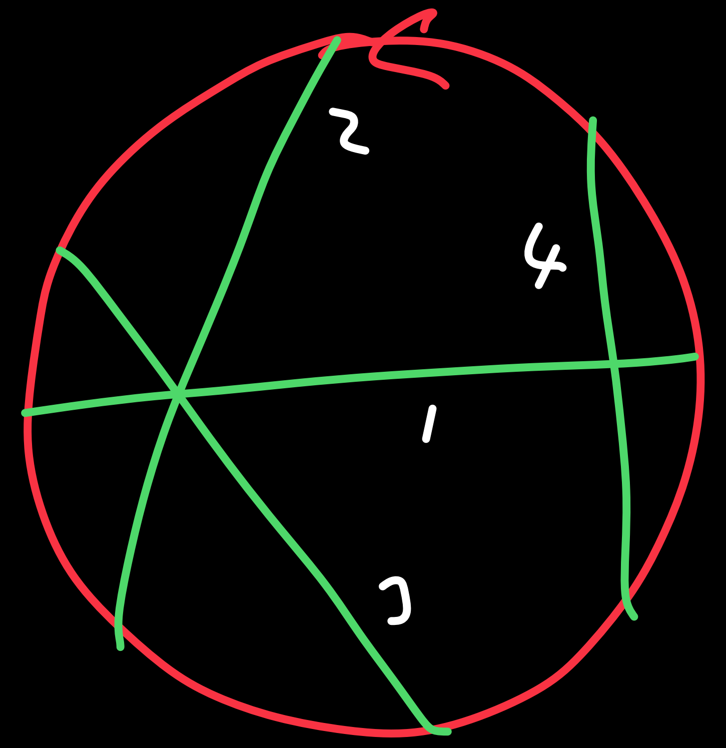
- For $\mathbb{G}$ a bouquet
- $\mathbf{IM}_{\mathbb{G}}$ is a 0-1-matrix
  = adjacency matrix of simple graph $I_{\mathbb{G}}$

$$\mathbf{IM}_{\mathbb{G}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$



- Thus $D(\mathbb{G})$ is completely determined by $I_{\mathbb{G}}$
- So $D(\mathbb{G}) = D(\mathbb{H})$
  $\iff I_{\mathbb{G}} = I_{\mathbb{H}}$ when $\mathbb{G}$ and $\mathbb{H}$ bouquets

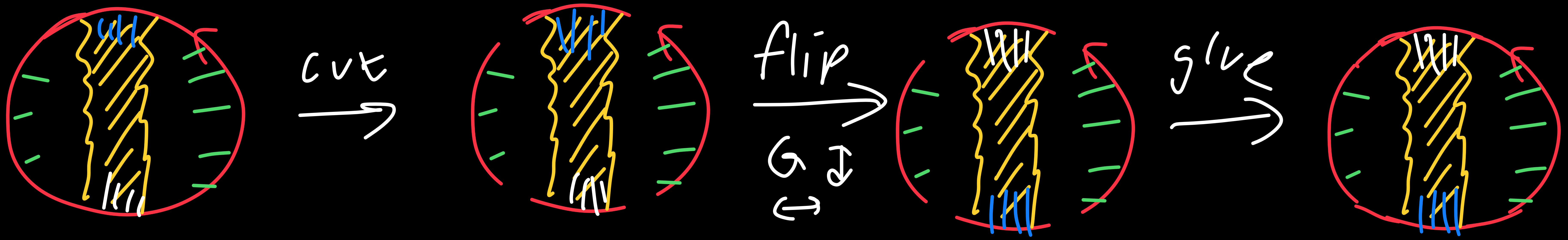




- But $I_{\mathbb{G}}$ is the intersection graph of a chord diagram
  i.e. it is a circle graph.
- It is known when two circle graphs arise from the same chord diagram.

Circle graphs

- It is known when two circle graphs arise from the same chord diagram: [Bouchet ’87; Gabor—Supowit—Hsu ’89; Chmutov—Lando ’07; Courcelle ’08]



- ⇝ when $\mathbb{G}$ and $\mathbb{H}$ bouquets:

$$D(\mathbb{G}) = D(\mathbb{H}) \iff I_{\mathbb{G}} = I_{\mathbb{H}} \iff$$ $\mathbb{G}$ and $\mathbb{H}$ are “mutants”

- What if $\mathbb{G}$ and $\mathbb{H}$ are *not* bouquets? (so they have more than one vertex)

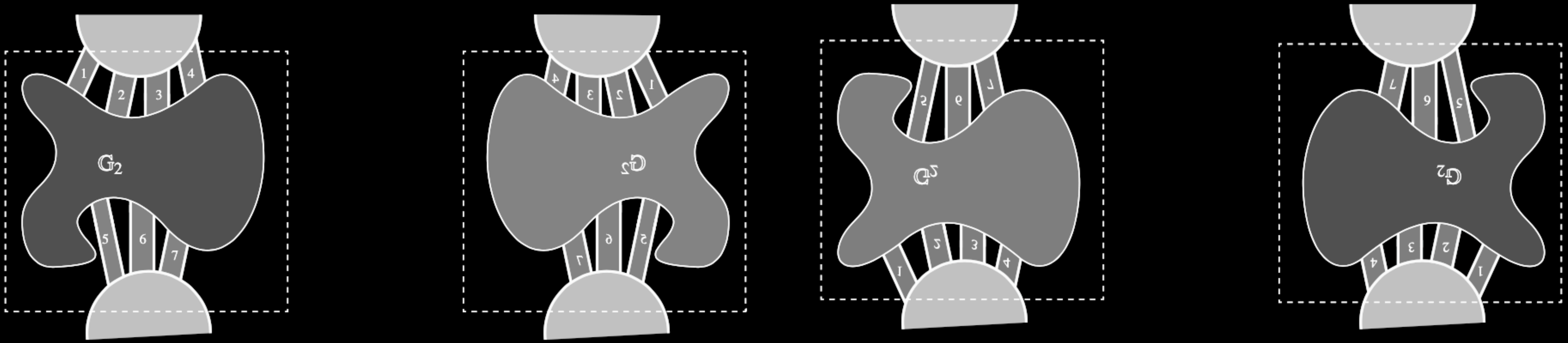
Applying to ribbon graphs

- $\mathbb{G}$ and $\mathbb{H}$ bouquets: $D(\mathbb{G}) = D(\mathbb{H}) \iff I_{\mathbb{G}} = I_{\mathbb{H}} \iff$ $\mathbb{G}$ and $\mathbb{H}$ are mutants

- What if $\mathbb{G}$ and $\mathbb{H}$ are *not* bouquets (so they have more than one vertex)?

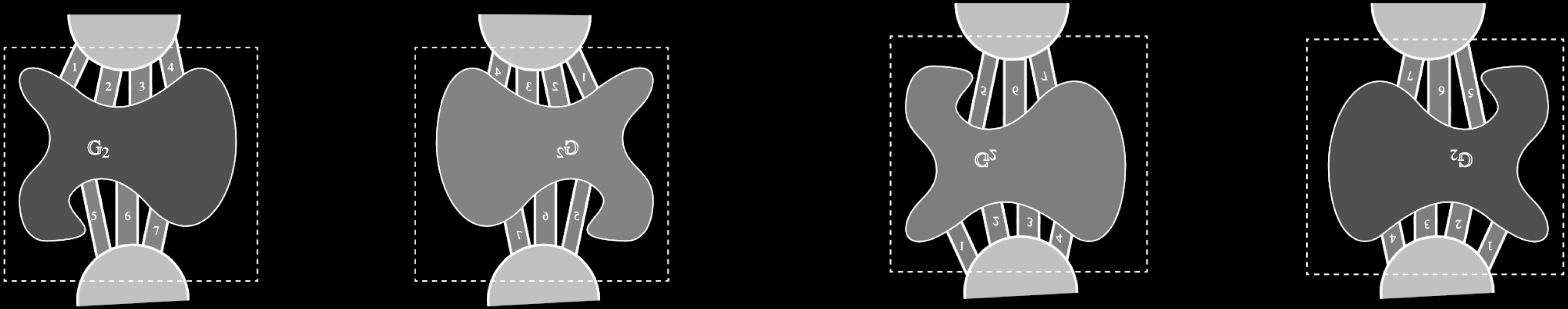
- Then pick subset A of edge so that $\mathbb{G}^A$ and $\mathbb{H}^A$ are bouquets:

- $D(\mathbb{G}) = D(\mathbb{H}) \iff D(\mathbb{G}^A) = D(\mathbb{H}^A) \iff I_{\mathbb{G}^A} = I_{\mathbb{H}^A} \iff \mathbb{G}^A$ and $\mathbb{H}^A$ are mutants $\iff$ $\mathbb{G}$ and $\mathbb{H}$ are related by "mutation":

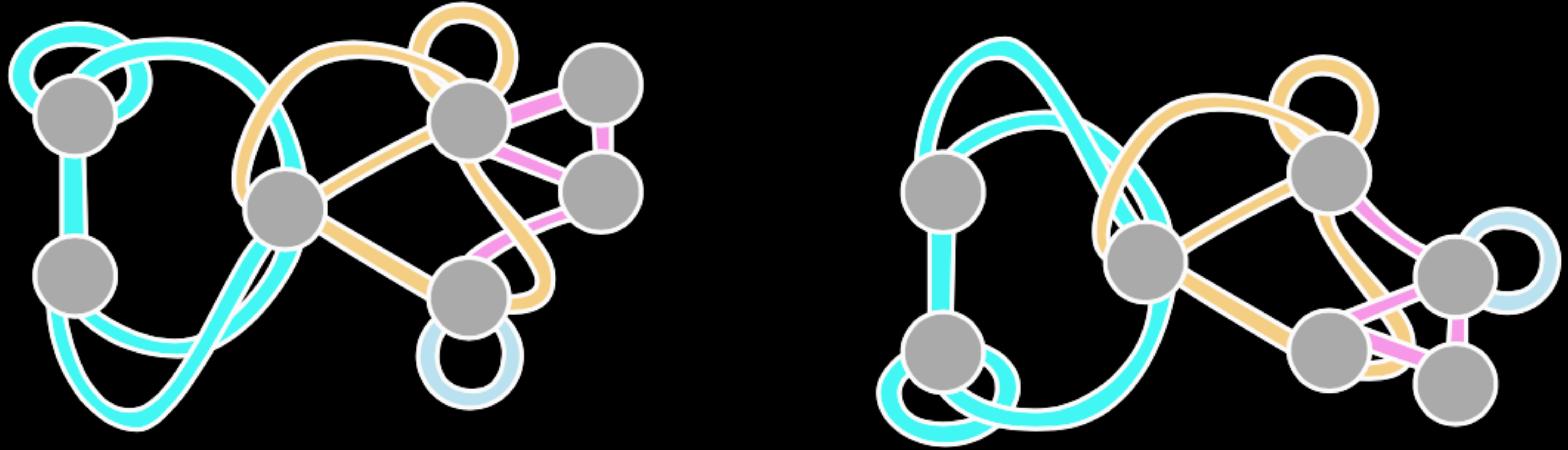


Ribbon graphs with the same delta-matroids

- [M—Oh '21] $D(\mathbb{G}) = D(\mathbb{H}) \Longleftrightarrow$ $\mathbb{G}$ and $\mathbb{H}$ are related by



- The quasi-trees determine the ribbon graph up to this move.



Back to where we started

- *Do the spanning trees determine the graph?*
- *Precisely, if you know the edge set of each spanning tree, and any loops in the graph, do you know the graph?*
- *Whitney's answer: yes up to 2-isomorphism.*
- *Do the spanning quasi-trees determine the embedded graph?*
- *Precisely, if you know the edge set of each spanning quasi-tree, and any loops in the graph, do you know the embedded graph?*
- *Yes, up to mutation.*
- *The point is that studying embedded graphs through their delta-matroids (quasi-trees) opens new door, just as studying graphs through their matroids does.*

Thanks!