THE γ-GRAPH OF A GRAPH

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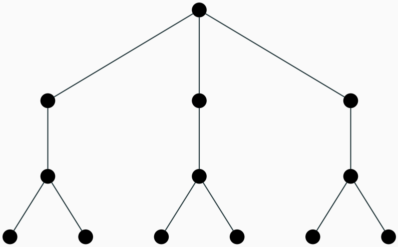
St Francis Xavier University

AGENDA

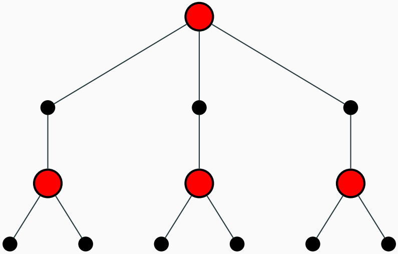
1. INTRODUCTION
2. THE γ-GRAPH OF TREES
3. SEQUENCES OF γ-GRAPHS
4. FUTURE WORK

INTRODUCTION

A dominating set



A dominating set



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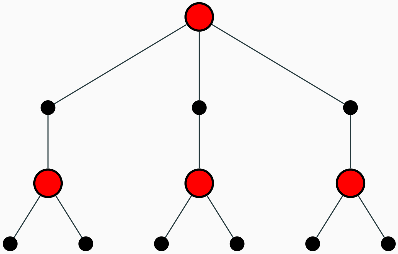
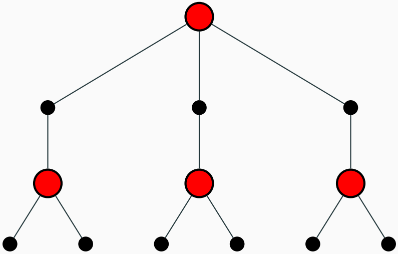


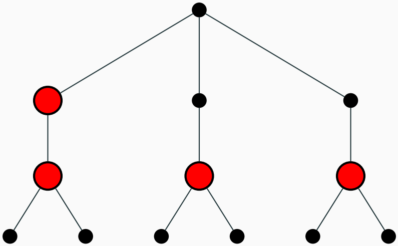
A dominating set

- A subset of vertices D is a *dominating set* of G if $N[D] = V(G)$, that is, every vertex not in D is adjacent to a vertex in D.
- The *domination number* of a graph G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G.
- A dominating set of minimum cardinality is said to be a γ-*set*.

γ-sets



γ-sets



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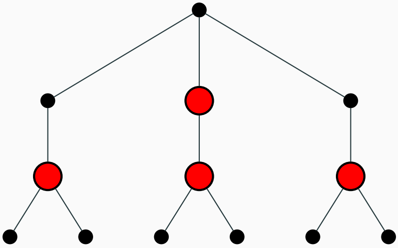

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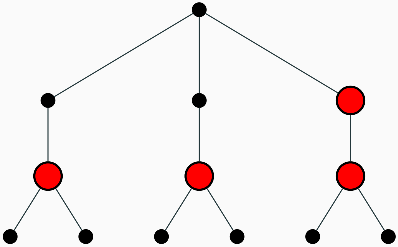
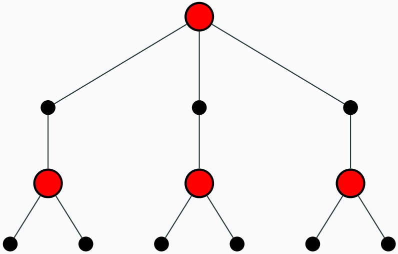


γ-graph of this tree



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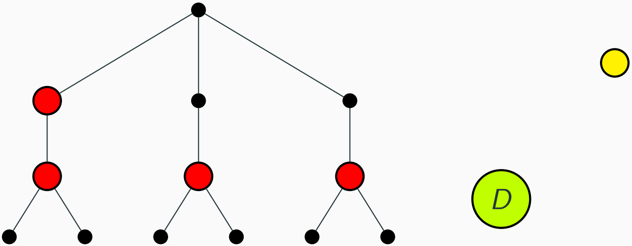

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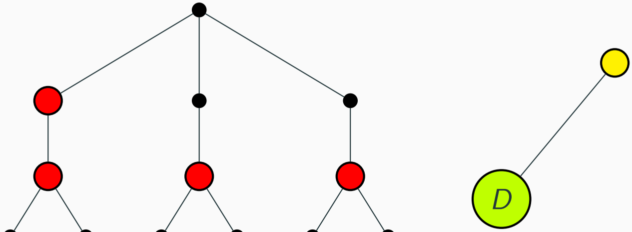

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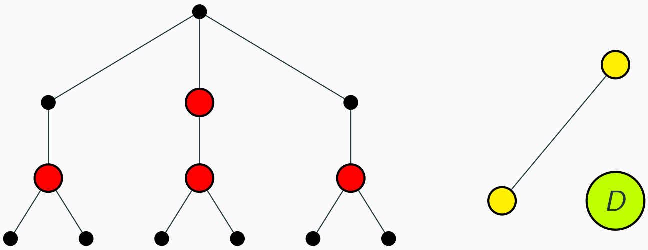
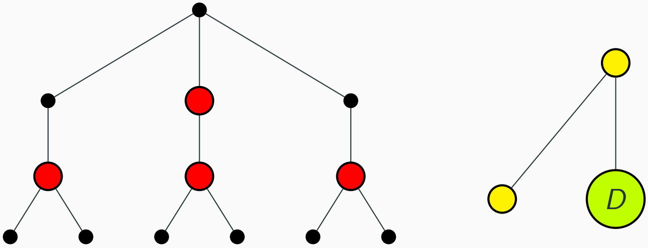


Slide γ-graph of this tree



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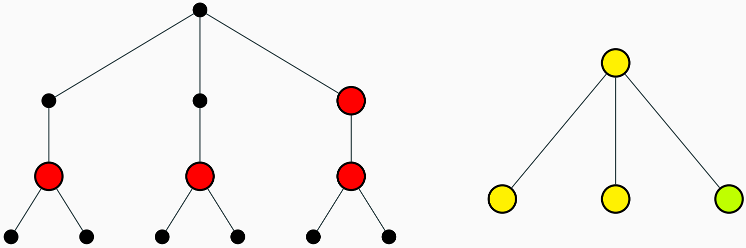

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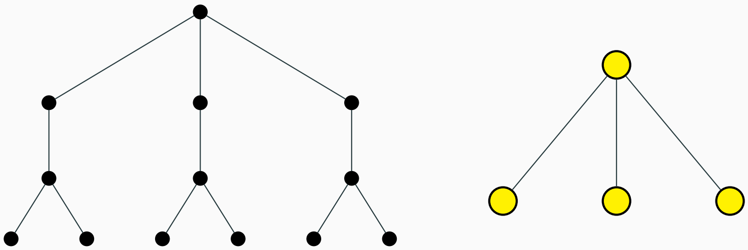
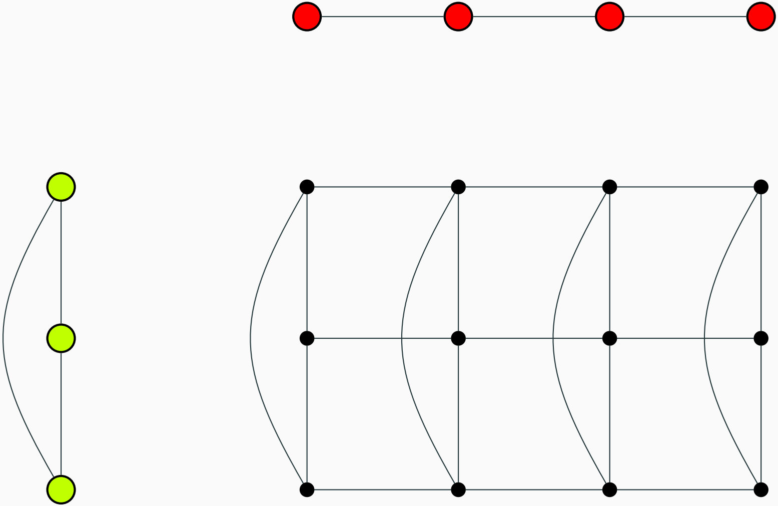


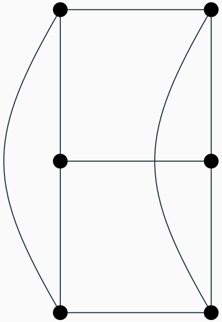
(Slide) γ-graph of a graph G

- Introduced by Fricke, Hedetniemi, Hedetniemi, Hutson in 2011.
- It is a *reconfiguration* graph denoted $G(\gamma)$ or $\mathcal{S}(G, \gamma)$.
- Vertex sets are the minimum dominating sets of G.
- Two γ-sets are adjacent if we can obtain one from the other by swapping adjacent vertices.

Cartesian Product ($K_3 \square P_4$)



γ-Graph of $K_3 \square K_2$



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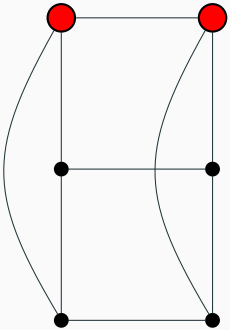

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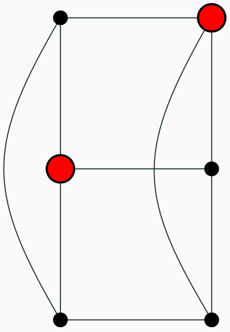

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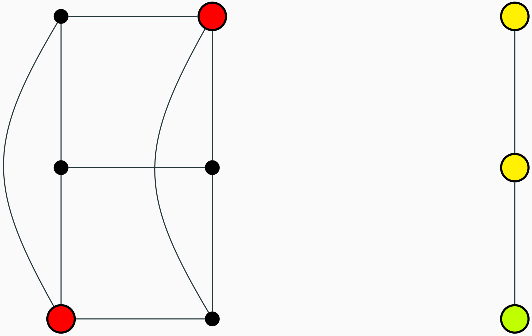

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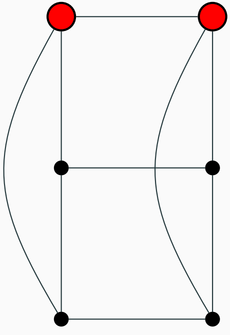

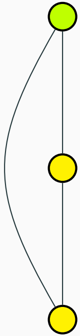

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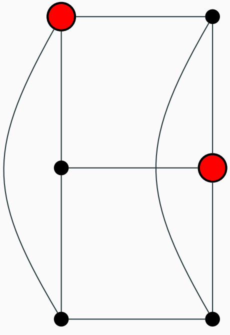

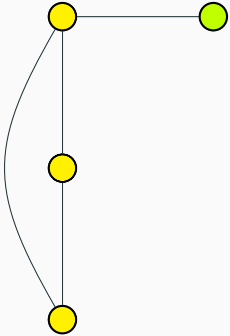

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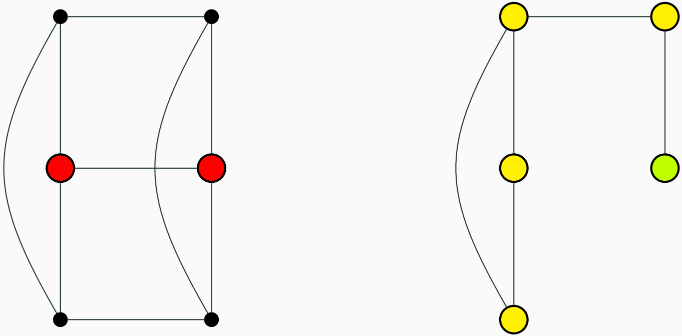
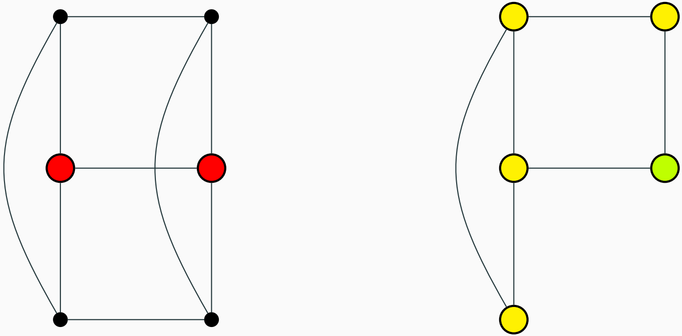


γ-Graph of $K_3\Box K_2$



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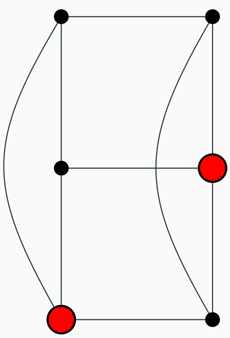

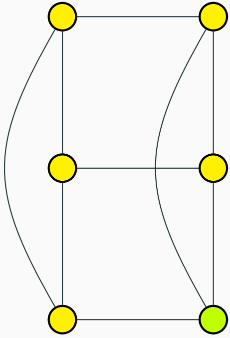

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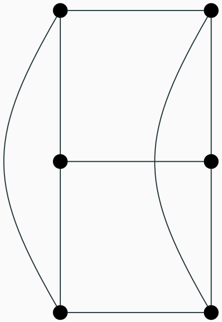

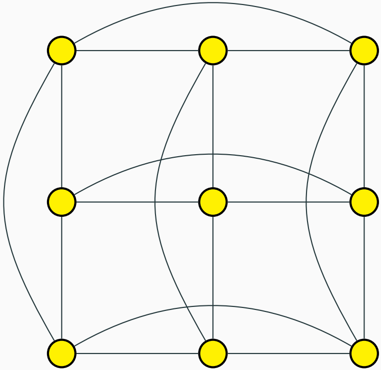

$$K_3 \square K_2 \xrightarrow{\gamma} K_3 \square K_3$$

(Slide) γ-graphs

- Jump γ-graphs were introduced by Subramanian and Sridharan
- Fricke, Hedetniemi, Hedetniemi, Hutson posed 7 questions in their original paper.

(Slide) γgraphs

1. Which graphs are γ-graphs? Can you construct a graph H that is not a γ-graph of any graph G?
2. Is $\Delta(T(\gamma)) = O(n)$ for every tree T of order n?
3. Is $diam(T(\gamma)) = O(n)$ for every tree T of order n?
4. Is $|V(T(\gamma))| \leq 2^{\gamma(T)}$ for every tree T?
5. For which graphs G is $G(\gamma) \cong G$?
6. Under what conditions is $G(\gamma)$ a disconnected graph?
7. Which graphs are γ-graphs of trees?

Which graphs are γ-graphs?

Theorem (Connelly, Hutson, and Hedetneimi)
For every graph G there is a graph H so that $H(\gamma) \equiv G$.

- Proof outline: Let H be the join of G with the disjoint union of a star and an independent set

(Slide) γgraphs

1. Which graphs are γ-graphs? Can you construct a graph H that is not a γ-graph of any graph G?
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Progress

The next three questions were solved by Edwards, MacGillivray, and Nasserasr:

Theorem (Edwards, MacGillivray, and Nasserasr)
If T is a tree of order n having s stems, then

1. $\Delta(T(\gamma)) \leq n - \gamma(T)$
2. $diam(T(\gamma) \leq 2(2\gamma(T) - s)$
3. $|V(T(\gamma))| \leq ((1+\sqrt{13})/2)^{\gamma(T)}$

Progress

The remaining three questions are all open:

5. For which graphs G is $G(\gamma) \cong G$?
6. Under what conditions is $G(\gamma)$ a disconnected graph?
7. **Which graphs are γ-graphs of trees?**

Other directions

- Mynhardt and Teshima discussed variations and provided further directions (2018)
- Lemańska and Zyliński have determined tight bounds on the diameters for $T(\gamma)$
- Survey of reconfiguration graphs by Mynhardt and Nasserasr (2021+)
- van Bommel showed $K_{2,3}$ is not the γ-graph of any bipartite graph.

THE γ-GRAPH OF TREES

Algorithm

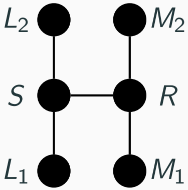
Root a tree at vertex of high degree

Recursively find the γ-graph in two cases

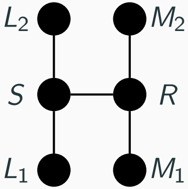
γ-Trees of Trees

Which trees are the γ-graphs of trees?

γ-Trees of Trees



γ-Trees of Trees



This is not the *gamma*-graph of any tree.

Theorem

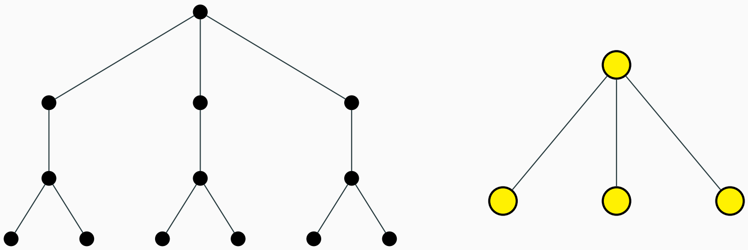
If G is a tree, then G is a γ-graph of some tree if and only if H is not a subtree of G.

γ-Trees of Trees

Theorem

If G is a tree, then G is a γ-graph of some tree if and only if H is not a subtree of G.

Slide γ-graph of this tree



Lemma

Let T be a tree and let $T(\gamma)$ be the γ-graph of T. Every edge of $T(\gamma)$ is a cut-edge or is contained in a 4-cycle.

Lemma

Let G be a γ-graph of a tree T with a cut-edge e, and let G_1 and G_2 be the components of $G \setminus e$. Then $G_1 + e$ and $G_2 + e$ are each γ-graphs of trees.

Cartesian Product and γ-Graphs of Trees

Theorem

Let T be a tree and let $T(\gamma)$ be the γ-graph of T. Then $T(\gamma)$ does not contain $K_{2,3}$ as a subgraph.

Cartesian Product and γ-Graphs of Trees

Theorem (Sabidussi(60), Vizing(63))

All finite connected graphs have a unique prime factorization with respect to Cartesian multiplication.

Theorem

Let G be a Cartesian product graph with $G = G_1 \Box G_2$, $G_i \neq K_1$. Then G is the γ-graph of a tree T if and only if each G_i is the γ-graph of a tree T_i.

SEQUENCES OF γ-GRAPHS

Infinite Sequence

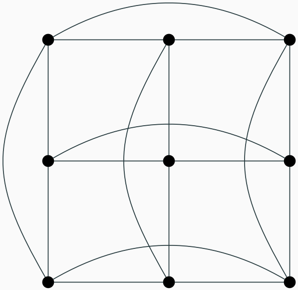
From the orginal paper:

Claim

$K_3 \square K_2 \xrightarrow{\gamma} K_3 \square K_3 \xrightarrow{\gamma} K_3 \square K_3 \square K_3 \xrightarrow{\gamma} \ldots.$

So it was concluded there is an infinite γ-graphs sequence.

Is $K_3 \square K_3 \xrightarrow{\gamma} K_3 \square K_3 \square K_3$?



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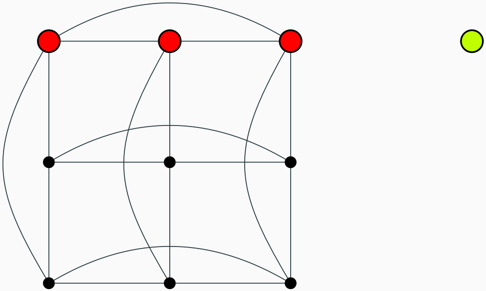

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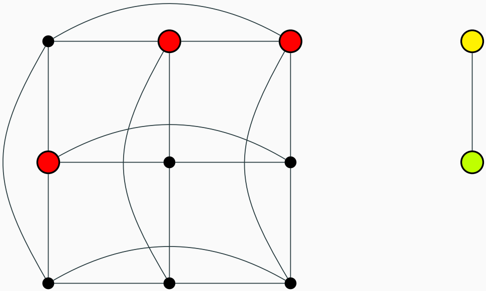

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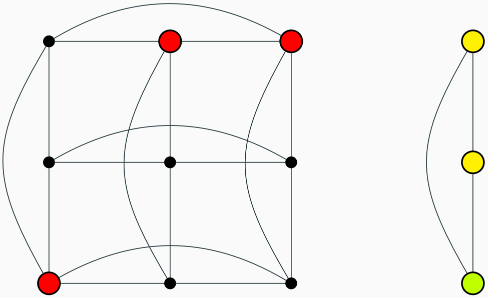

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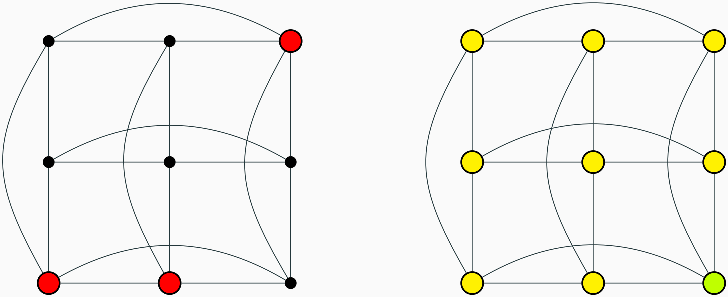

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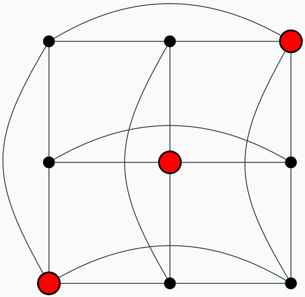

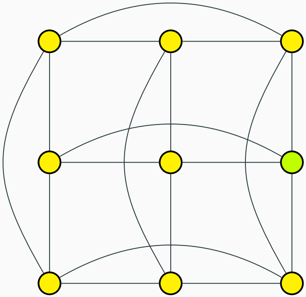

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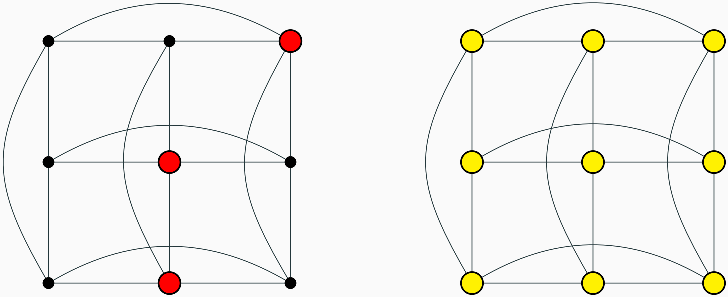

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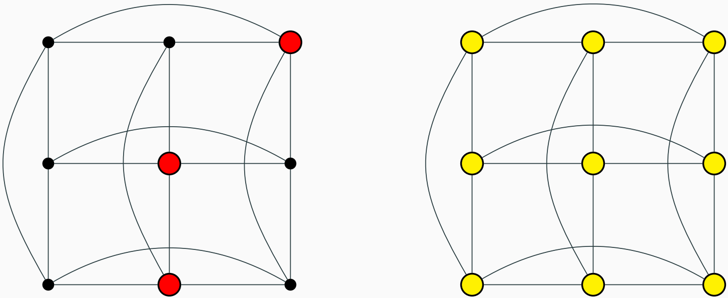
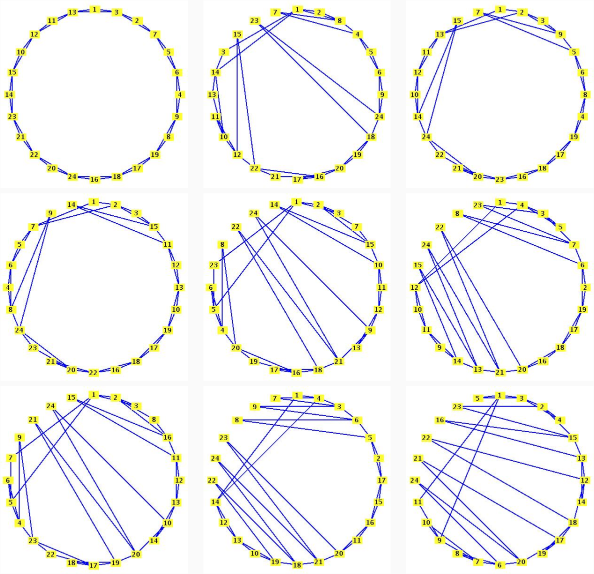


$$K_3^3 \dagger K_3^3$$

γ-graph of $K_3^3 \dagger K_3^3$



γ-graph Sequences

- We can show

$$K_3 \Box P_2 \xrightarrow{\gamma} K_3 \Box K_3 \xrightarrow{\gamma} K_3^3 \dagger K_3^3 \xrightarrow{\gamma} \mathbf{G} \xrightarrow{\gamma} \overline{K_{512}} \xrightarrow{\gamma} K_1$$

- It is an open question as to whether an infinite γ-graph sequence exists?
- However, there are arbitrarily long γ-graph sequences of paths.

FUTURE WORK

Future Work

5. For which graphs G is $G(\gamma) \cong G$?
6. Under what conditions is $G(\gamma)$ a disconnected graph?
7. Which graphs are γ-graphs of trees?
8. Does there exists a γ-graph sequence of finite graphs that grows without bound, or if every γ-graph sequence of finite graphs eventually repeats.

THANK YOU