Robustness of Complex Networks

presentation at

ATLANTIC GRAPH THEORY SEMINAR

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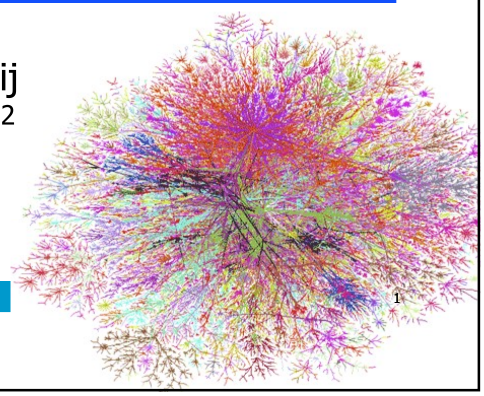
19 January 2022
TU Delft



1

About me

TU Delft

1988 - 1993

1994 - 1996

kpn

Royal Dutch Telecom

1997 - 2003

Applied → TNO

SUTD SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

Theoretical → TU Delft

I'm Back

2018 - 2019

2005 - now

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Robustness of Complex Networks

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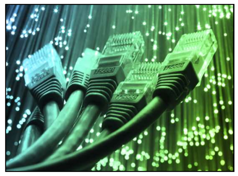
Introduction (1/2)

- Society is critically depending on complex networks



- Robustness: extent to which a complex network can cope with disruptions
  - failures of its nodes and/or links
- **Use graph theory to deal with robustness**

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Introduction (2/2)

- How to quantify network robustness?
- What part of the network is most vulnerable?
- How to make the network more robust?



Critical Infrastructures

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Outline

- Terminology
- Robustness w.r.t. malware spread
- Robustness of a gas distribution network
- Robustness of network controllability
- Wrap-up
- Bonus

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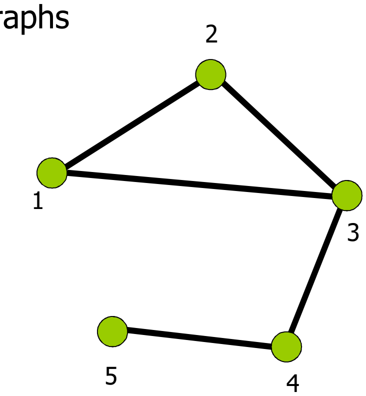
Terminology (1/4)

- Networks are represented as graphs
- Graph G(N,L)
  - N = number of nodes
  - L = number of links
- Graphs can be
  - undirected or directed
  - unweighted or weighted



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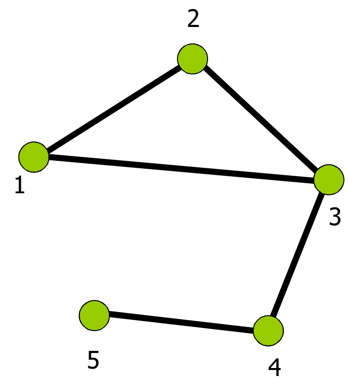
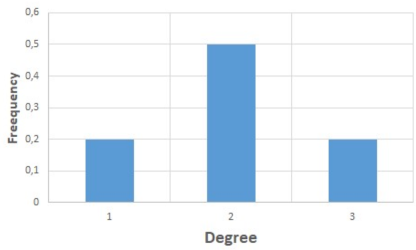
Terminology (2/4)



- degree D_i of node i
  - number of neighbours of node i
- degree distribution



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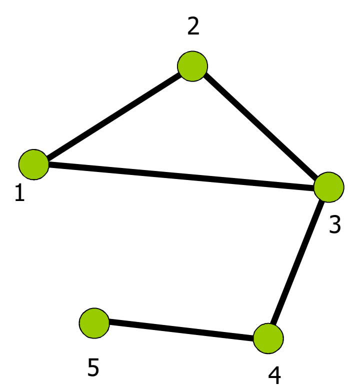
Terminology (3/4)

- Adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$



- ρ = spectral radius = largest eigenvalue of A

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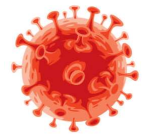
Terminology (4/4)

- The objects we study are NOT static
- Dynamical processes ON network
    -



- Network elements subject to stochastic process

- Methods from statistical physics
    - Mean-field approach
    - Simulations vs. models

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Robustness w.r.t. malware spread (1/10)

- Spread of malware (malicious software)

- Relation malware spread and network structure?

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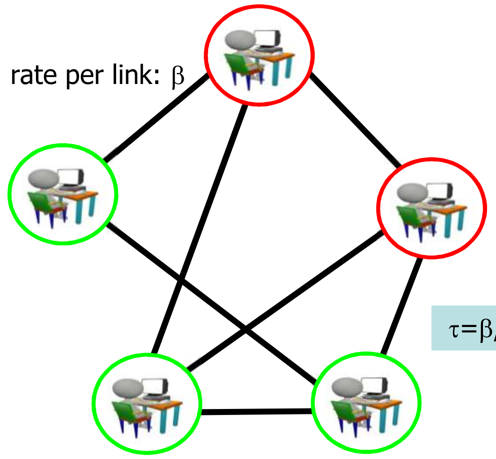
Robustness w.r.t. malware spread (2/10)

Infection rate per link: β



Curing rate per node: δ

τ=β/δ: effective spreading rate

Susceptible-Infected-Susceptible (SIS) model

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Robustness w.r.t. malware spread (3/10)

- Epidemic threshold τ_c
  - Effective spreading rate ≤ τ_c → malware dies
  - Effective spreading rate > τ_c → malware survives

$$\tau_c = \frac{1}{spectral\ radius}$$

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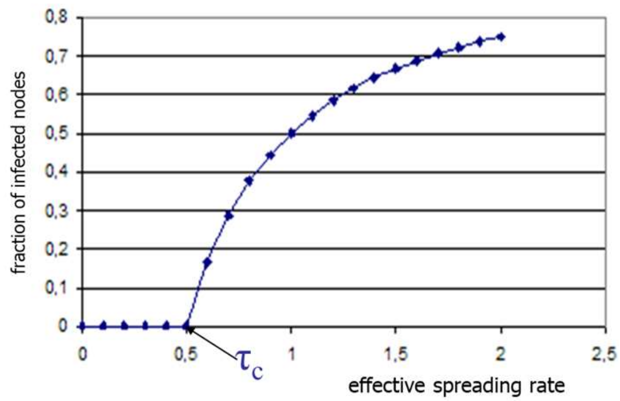
Robustness w.r.t. malware spread (4/10)



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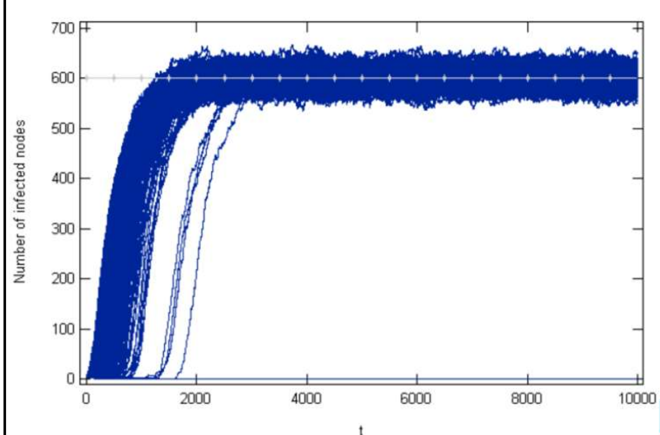
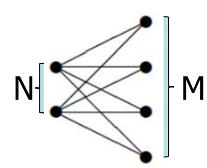
Robustness w.r.t. malware spread (5/10)

- Complete bi-partite graphs: $K_{N,M}$





$$\tau_c = \frac{1}{\sqrt{MN}}$$

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$K_{10,990}$: $\tau = 0.15 > 0.0101 = \tau_c$

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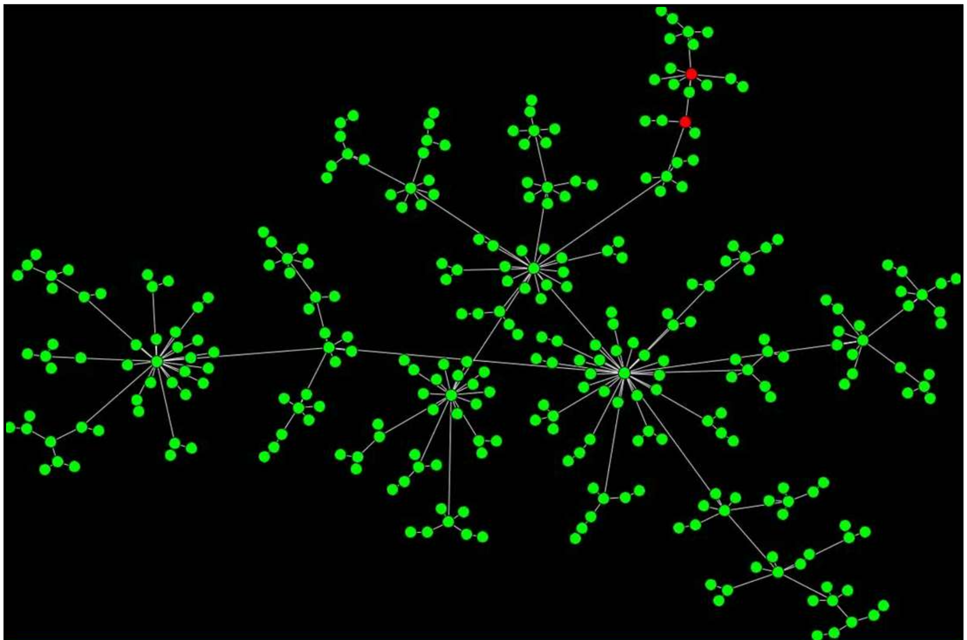

Robustness w.r.t. malware spread (7/10)

- smaller ρ: more robustness against malware spread
- connected graphs: which topology has the smallest ρ ?
  - the path P_N 

$$\rho(P_N) = 2\cos(\frac{\pi}{N+1})$$

- what if we pose extra conditions?

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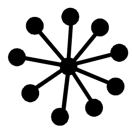
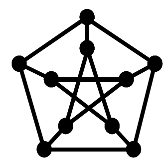
Robustness w.r.t. malware spread (8/10)

- Relation between minimal ρ and diameter of a graph?

- Graphs on N nodes with diameter 2:
  Minimal $\rho = \sqrt{N-1}$
  - Star topology
  - 3 additional cases : regular graphs (N = 5, 10, 50)



N =10



ρ = 3

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Robustness w.r.t. malware spread (9/10)

- Found minimal ρ for *Diameter* $\in$ {⌊N/2⌋,N-3, N-2, N-1}

- And for nearly all graphs on at most 20 nodes

$D \setminus n$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
2		1.4142	1.7321	2	2.2361	2.4495	2.6458	2.8284	3	3.1623	3.3166	3.4641	3.6056	3.7417	3.8730	4	4.1231	4.2426	4.358
3			1.6180	1.8478	2	2	2.3028	2.2784	2.4728	2.4860	2.5616	2.6970	2.7817	2.7321	2.8779	$\leq$ 2.9755	$\leq$ 3	$\leq$ 3.0742	$\leq$ 3
4				1.7321	1.9021	2	2	2	2.2361	2.2361	2.2230	2.3686	2.3778	2.3989	2.4303	2.5335	$\leq$ 2.7024	$\leq$ 2.7498	$\leq$ 2.79
5					1.8019	1.9319	2	2.0840	2	2	2.1701	2.2105	2.1987	2.1907	2.3028	2.3167	2.3228	2.3536	$\leq$ 2.54
6						1.8478	1.9499	2	2.0743	2.0743	2	2	2.1463	2.1940	2.1829	2.1753	2.1701	2.2688	$\leq$ 2.33
7							1.8794	1.9616	2	2.0684	2.1067	2.0684	2	2	2.1285	2.1693	2.1723	2.1649	2.15
8								1.9021	1.9696	2	2.0647	2.1010	2.1010	2.0647	2	2	2.1149	2.1505	2.16
9									1.9190	1.9754	2	2.0623	2.0912	2.1149	2.0912	2.0623	2	2	2.10
10										1.9319	1.9796	2	2.0608	2.0840	2.1120	2.1120	2.0840	2.0608	2
11											1.9419	1.9829	2	2.0598	2.0785	2.1054	2.1183	2.1054	2.07
12												1.9499	1.9854	2	2.0592	2.0743	2.1010	2.1169	2.11
13													1.9563	1.9874	2	2.0588	2.0710	2.0981	2.11
14														1.9616	1.9890	2	2.0586	2.0684	2.09
15															1.9659	1.9904	2	2.0584	2.06
16																1.9696	1.9915	2	2.05
17																	1.9727	1.9924	2
18																		1.9754	1.99
19																			1.97

Minimal ρ for *Diameter* = 3?

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Robustness w.r.t. malware spread (10/10)

Virus spread in networks P Van Mieghem, J Omic, RE Kooij
IEEE/ACM Transactions On Networking 17 (1), 1-14, 2009

The minimal spectral radius of graphs with a given diameter
ER van Dam, RE Kooij
Linear Algebra and its Applications 423 (2-3), 408-419, 2008

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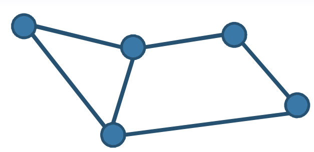
Robustness of a gas distribution network



- N nodes
- L links
- undirected graph

- Network availability = Pr {network is **connected**}
  - Nodes always operational
  - Each link interdependently operational with probability *p*
  - All-terminal reliability

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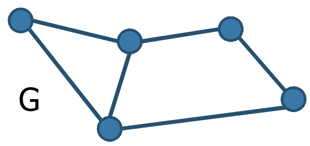
Reliability polynomial



$R_G(p)$ = Pr {G is connected}

$$R_G(p) = F_0 p^L + F_1(1-p)p^{L-1} + F_2(1-p)^2 p^{L-2} + .. + F_{L-N+1}(1-p)^{L-N+1} p^{N-1}$$

F_i: # of sets of i links, whose removal leave G connected

F_1 = 6

A case study

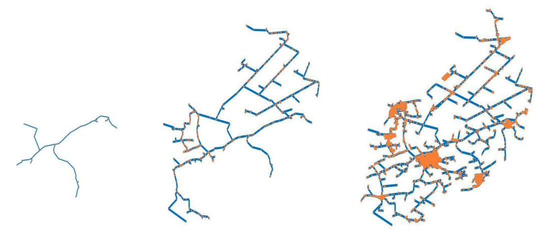
- Links: gas pipes
- Nodes: points where pipes connect

Network details

7240 households

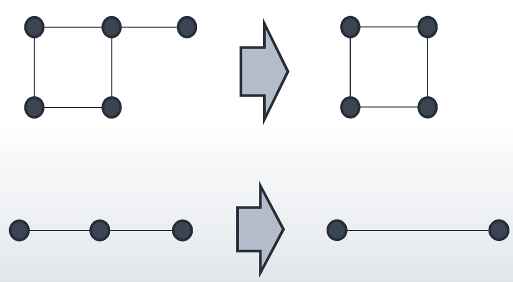
- One entry point for gas
- Network consists of three parts
  - 8, 3 and 0.1 bar

Network	8 bar	8 & 3 bar	Full network
Nodes	256	1845	20567
Links	255	1851	20749



Reductions on the network

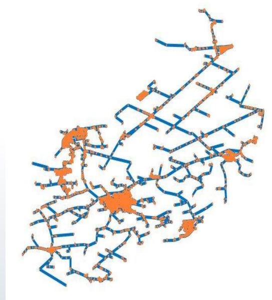
- Network is too large to process
- Reduce its size without loss of relevant information

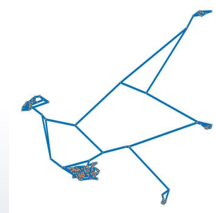


Shooman (1995)

Full network: significant reduction

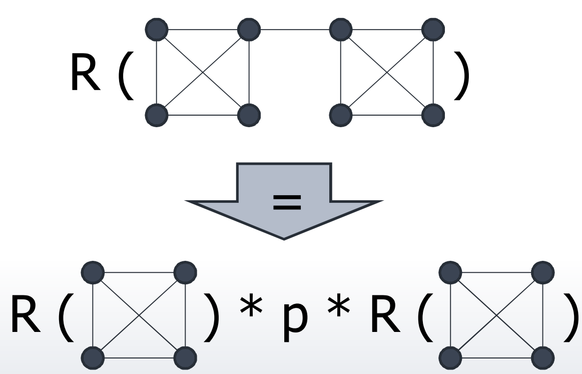
	Before	After
Nodes	20567	262
Edges	20749	393





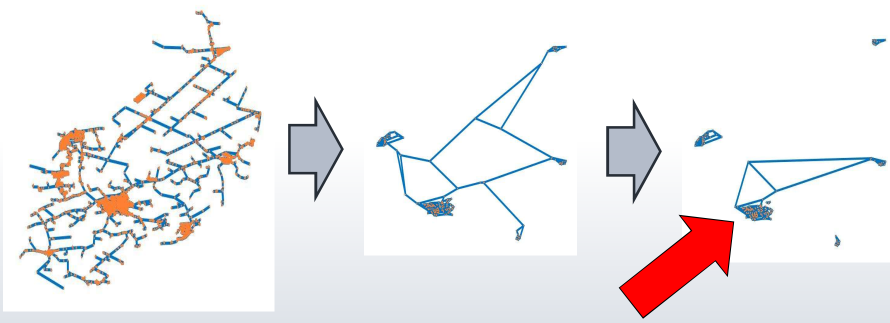
- Still too large for direct computation

Additional reductions



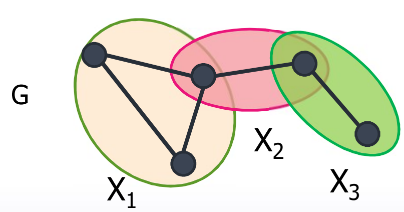
Result of split: 5 sub-networks

	Before	Sub 1	Sub 2	Sub 3	Sub 4	Sub 5
Nodes	20567	34	12	186	4	12
Edges	20749	51	18	279	6	18



Largest sub-network: decomposition

- Decomposition based upon **pathwidth** of graph



- Computation time polynomial in pathwidth(G)

Results

Can we compute the exact availability of our gas network?

– Computation takes about **2 minutes**
– Individual p values depend on
  - Soil type
  - Length of pipes

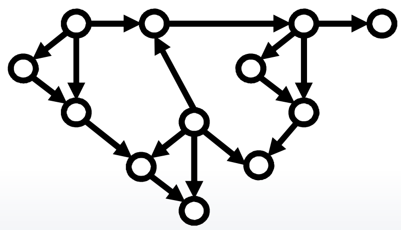
– Availability = 0.9919
  - 70 hours per year at least one node is disconnected
  - Assume every non-availability influences 3 households
  - Mean gas outage per household: 70*3600*3/7240 ≈ 104 seconds

Robustness of a gas distribution network

The reliability of a gas distribution network: A case study
W Pino, D Worm, R van der Linden, R Kooij
2016 International Conference on System Reliability and Science (ICSRS), 122-129

Robustness of network controllability

- Directed networks



- number of nodes = *N*
- number of links = *L*

Introduction to network control

$$\frac{d\boldsymbol{x}(t)}{dt} = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$$

$\mathbf{x}(t) = (x_1(t),\ldots\ldots,x_N(t))^T$: state of system at time t

$\mathbf{u}(t) = (u_1(t),\ldots\ldots,u_M(t))^T$: control input vector

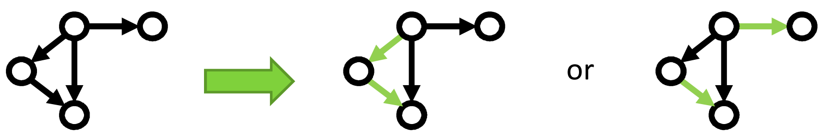
A: *NxN* matrix, describing systems connections

B: *NxM* input matrix, identifying nodes under outside control

- What is the minimum number of nodes that need to be controlled, to bring the system to a desired state?

Introduction to network control

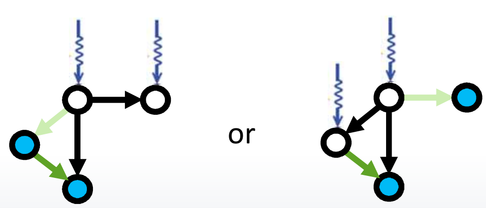
- How to find minimum number of driver nodes N_D?
- Through 'maximum matching' of network
  - maximum set of links that do not share start or end nodes



- Number of links in maximum matching is unique
- Maximum matching itself is NOT unique
- $O(N^{1/2}L)$ algorithm (Hopcraft-Karp) to find maximum matching

Introduction to network control

- Matched links point to matched nodes
- N_D = number of unmatched nodes



$N_D = 2$

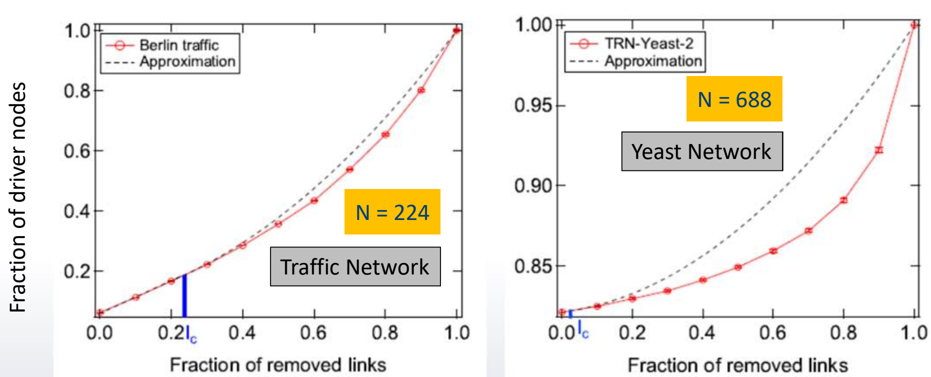
- Critical link: appears in every maximum matching
- l_C = fraction of critical links

Robustness of network controllability

- Assume links are removed from network
  - Random removal (failures)
  - Targeted removal (attacks)
- Number of driver nodes N_D will increase
- Analytic approximations for the increase in N_D
- Approximation
  - fraction of removed links $\leq l_C$: N_D linear in fraction of removed links
  - fraction of removed links $> l_C$: N_D quadratic in fraction of removed links

Robustness of network controllability

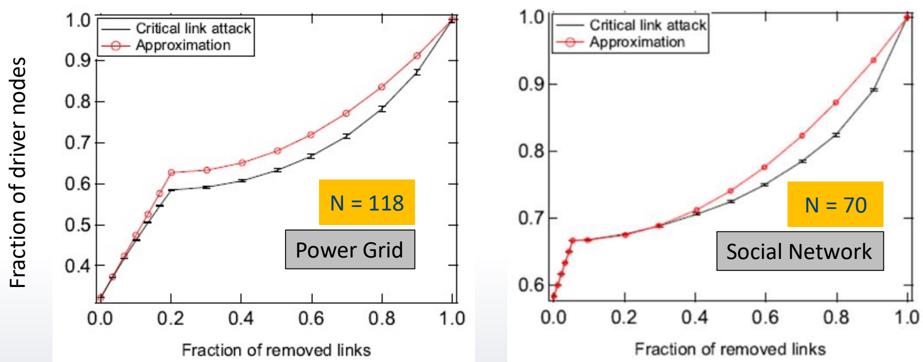
- Random link removal



Good results for small number of removals

Robustness of network controllability

- Targeted link removal



- Good results for small number of removals
- Approximation is worst-case

Robustness of network controllability

Quantifying the robustness of network controllability
P Sun, RE Kooij, Z He, P Van Mieghem
2019 4th International Conference on System Reliability and Safety

Wrap-up

- Robustness of complex networks
- Societal relevance
- Quantification of robustness
  - Malware spread
  - Availability in gas distribution network
  - Network controllability
- Methods from statistical physics

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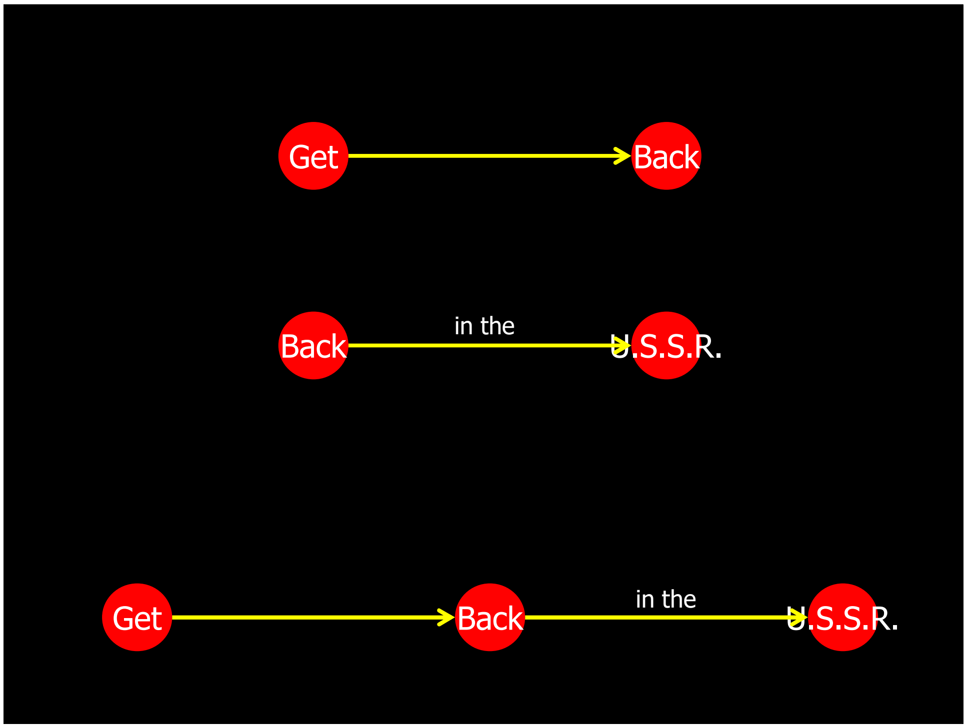
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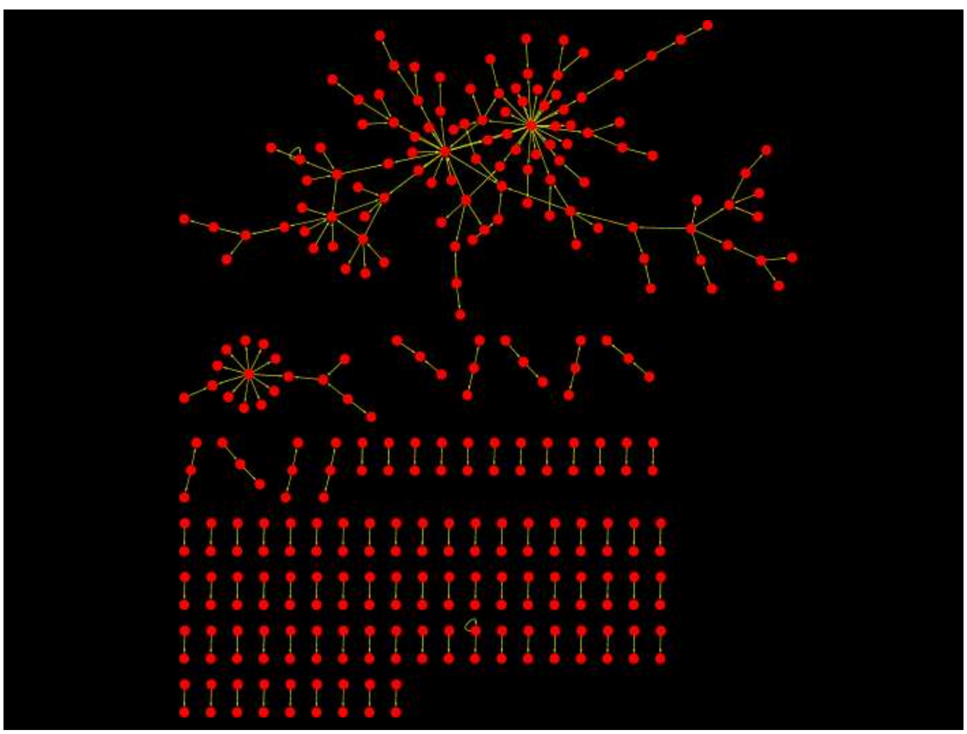

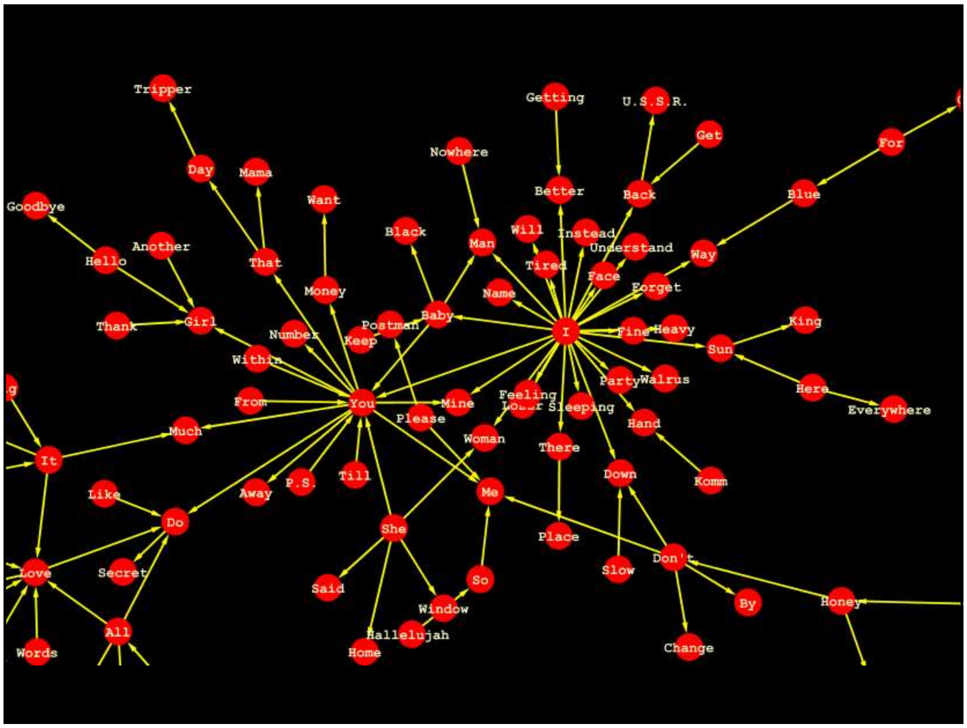
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Another
Hello
That
Tired
Face
Money
Name
Forget
Baby
Thank
Girl
Postman
Number
Keep
I
Fine
Heavy
King
Sun
Within
Party
Walrus
Here
Feeling
From
You
Mine
Sleeping
Everywhere
Please
Hand
Much
Woman
There
It
Down
Komm
Like
P.S.
Till
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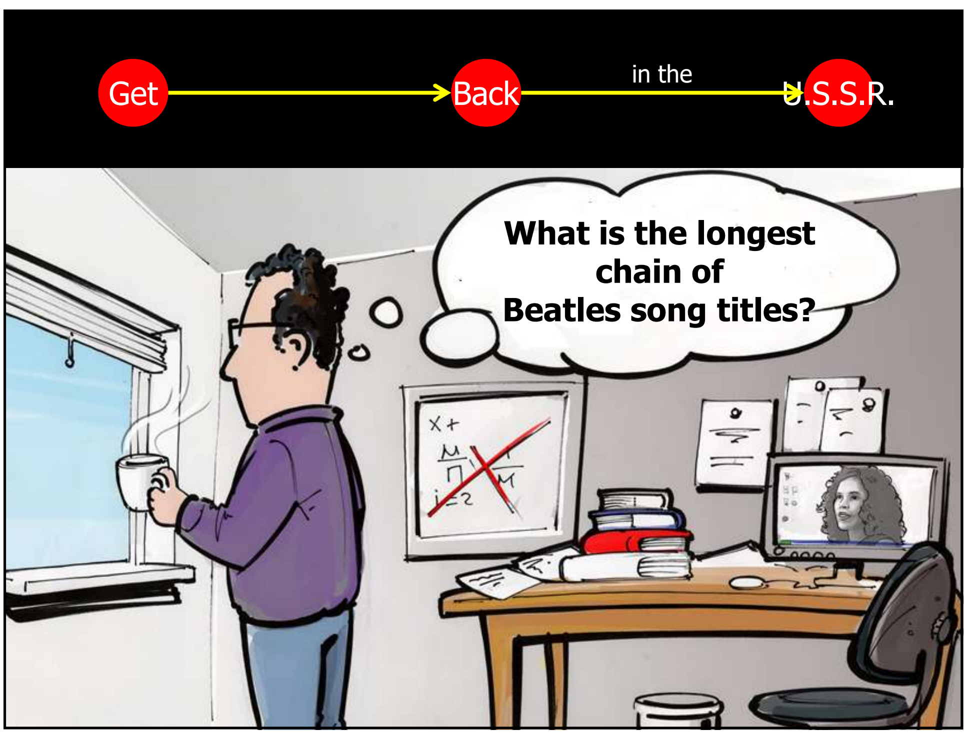
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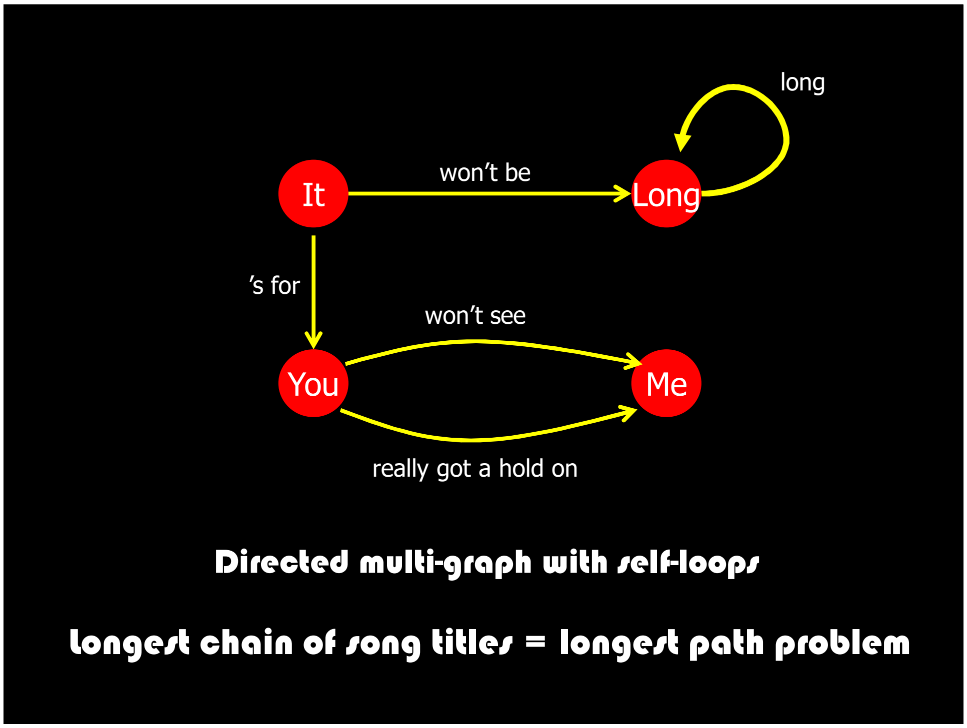
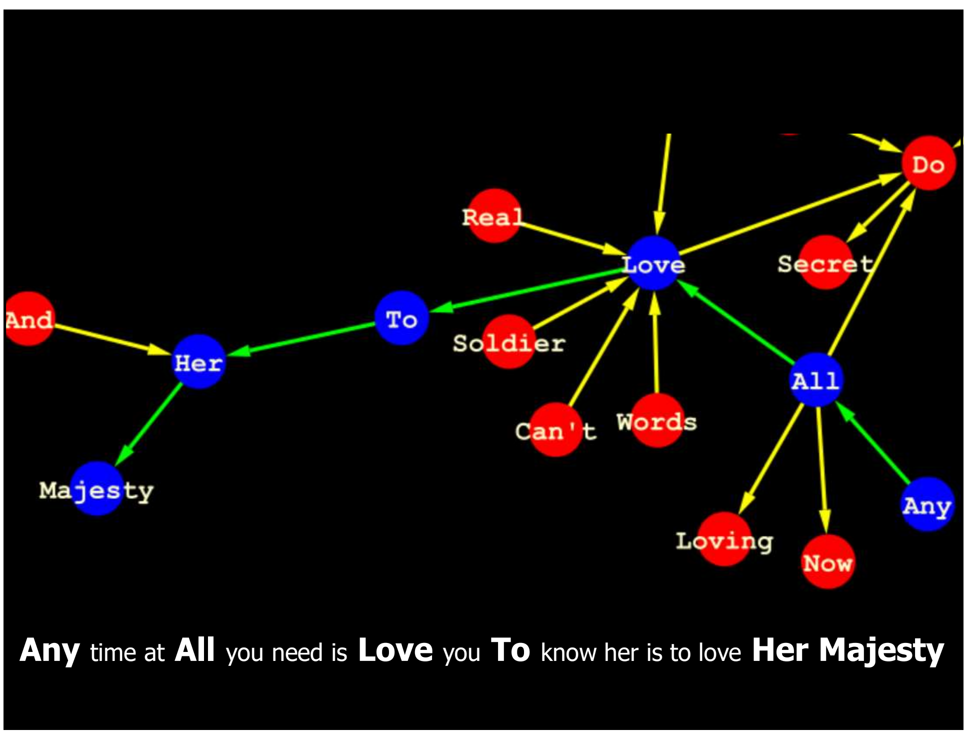
U.S.S.R.
What is the longest chain of Beatles song titles?


long
It
won't be
Long
's for
won't see
You
Me
really got a hold on
Directed multi-graph with self-loops
longest chain of song titles = longest path problem


Do
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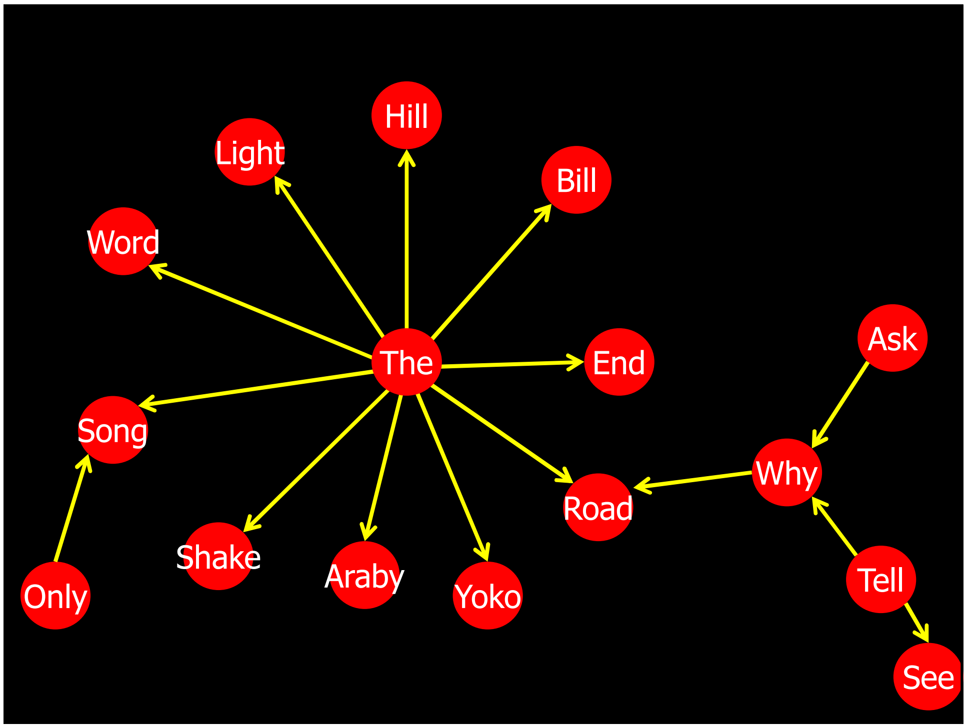
Any time at All you need is Love you To know her is to love Her Majesty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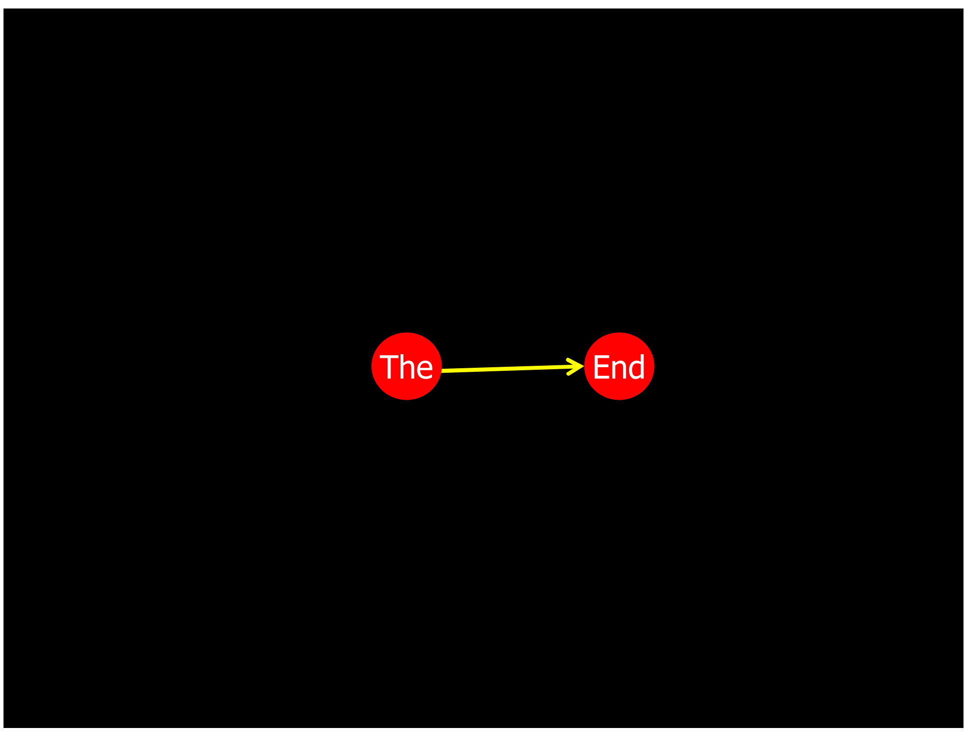
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