# Dynamical algebraic combinatorics and independent sets of graphs

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www.lacim.uqam.ca/~hugh/independent/independent.pdf
arXiv:1805.00815 (and arXiv:1712.10123 for background)

# Outline

# Dynamical algebraic combinatorics

Dynamical algebraic combinatorics has developed over the past ten years or so. Jim Propp and Tom Roby are two of the main developers; Jessica Striker and Nathan Williams have also been influential.

The basic idea is to consider:

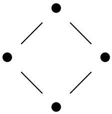
- an interesting (finite) set $E$, and
- an interesting permutation $\pi$ of $E$ (typically given as an algorithm).

We think of the permutation $\pi$ as describing one time-step of a time evolution. The idea is to ask questions about $(E, \pi)$ which are inspired by dynamical systems.

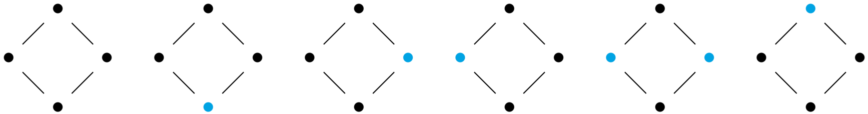
- What are the sizes of the orbits?
- Are there interesting functions on $E$ whose average value on every orbit is the same? (There is automatically a whole vector space of such functions; the question is whether there are functions which are interesting given $E$.) Such a function is said to exhibit homomesy (from the Greek for same average).
- The cyclic sieving phenomenon of Reiner, Stanton, and White can also be viewed as falling within this topic, though it predates the name.

# Rowmotion: the set

Let $P$ be a poset. Let $A(P)$ be the set of antichains (pairwise incomparable elements). We will take this as our ground set (what I was calling $E$).
For example, $P$ could be the product of two chains of length 2:



Its antichains are the following, where the elements of the antichains are marked in blue:



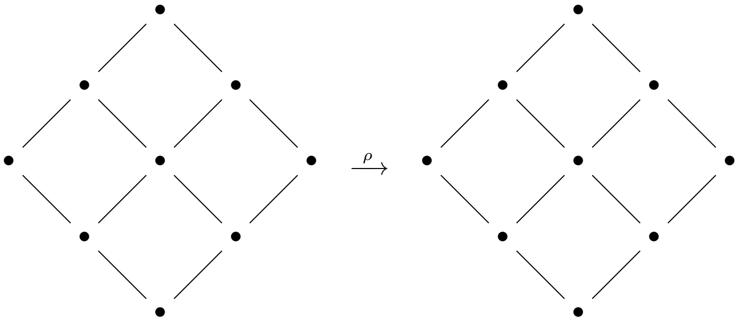
Note that antichains are exactly the independent sets of the comparability graph of $P$ (the directed graph where there is an edge from $x$ to $y$ if $x > y$). This will be important later.

# Rowmotion: the permutation

Rowmotion ($\rho$) is a permutation on the set $A(P)$ of antichains.

- Given an input $A \in A(P)$.
- Let $A^-$ be the corresponding down-closed set (order ideal): all elements below any elementof $A$.
- Return the minimal elements of $P \setminus D(A)$.

Since these steps are each reversible, it is indeed a permutation.
Examples on the product of two chains of length 3 (live action)!



# Rowmotion in an $a \times b$ grid

Rowmotion has been discovered several times; the earliest mention seems to be Brouwer and Schrijver in 1974. It has also been named several times; the name "rowmotion," from an influential paper by Striker and Williams, seems to have stuck. I will get back to why it is called rowmotion in a bit.

The first question we wanted to ask about such an action is: what are the orbit sizes?

It turns out that in an $a \times b$ grid, the orbit lengths all divide $a + b$.

In fact, there is an equivariant bijection from $(A(a \times b), \rho)$ to the rotation action on strings consisting of $a$ 1's and $b$ 0's. This rotation action obviously has period $a + b$. (This is a fairly easy exercise. It goes back to Brouwer and Schrijver.)

It also turns out that there is a nice homomesic statistic: the average size of an antichain is constant on all orbits; it is $ab/(a + b)$ (Propp–Roby).

# Rowmotion on other posets

You might well ask about rowmotion on other posets. Brouwer and Schrijver did so, and noticed that, in general, it was very bad. On a Boolean lattice, for example, the sizes of the orbits are a mess.

However, there are some other posets for which it is well-behaved, and the dynamical algebraic combinatorics community has been quite interested in studying these. The product of two chains is an example of a *minuscule poset*, and other minuscule posets also exhibit nice behaviour. There are other posets coming from Lie theory which exhibit nice behaviour for rowmotion as well.

I still haven't explained why rowmotion is called that, though, and since this will be important when we come to generalizing rowmotion to independent sets, I would like to do that now.

In order to do that, instead of thinking of the ground set of $\rho$ as the antichains of $P$, I will need to think of it as $J(P)$, the set of down-closed sets of $P$.

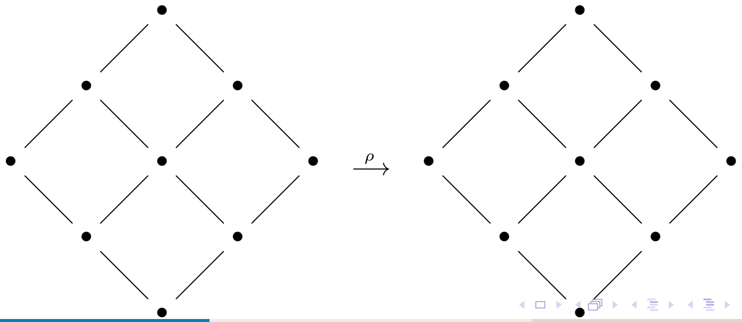
# Rowmotion on order ideals

Antichains and down-closed sets are naturally in bijection (the downset corresponding to $A$ is what I called $A^-$: it consists of all elements of $P$ below any element of $A$). Thus we can transport $\rho$ to a permutation of downclosed sets.

In fact, the definition is just as natural there:

- Start with $J \in J(P)$ a down-closed set.
- Let $M$ be the minimal elements of $P \setminus J$.
- Return $M^-$.

Examples on the product of two chains of length 3 (live action)!

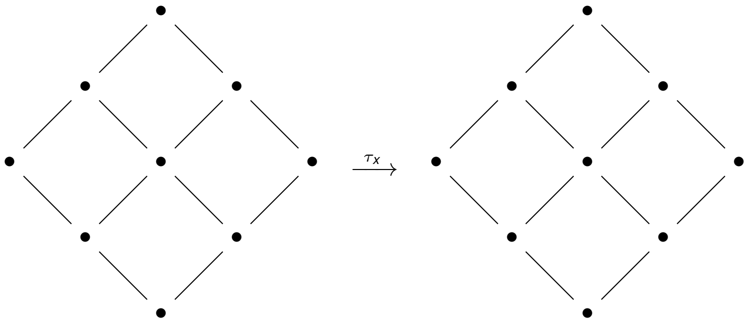


# Toggles on order ideals

For $x \in P$, define the toggle at $x$, written $\tau_x$, to be the involution on $J(P)$ defined by

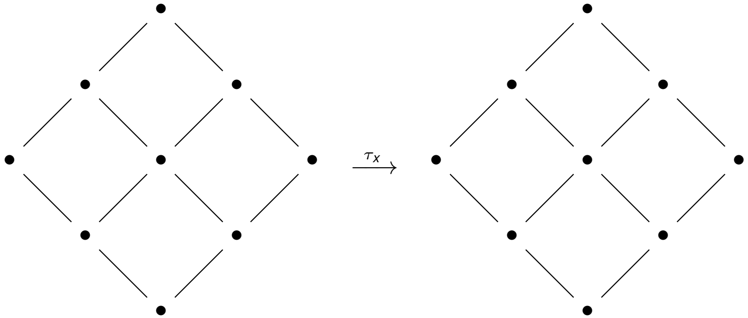
$$\tau_x(J) = \begin{cases} J \cup \{x\}, & \text{if } x \notin J \text{ and } J \cup \{x\} \in J(P) \\ J \setminus \{x\}, & \text{if } x \in J \text{ and } J \setminus \{x\} \in J(P) \\ J, & \text{otherwise} \end{cases}$$

Examples on the product of two chains of length 3 (live action)!



# Rowmotion via toggles

It is a theorem which goes back to Cameron and Fon-der-Flaass in 1995 that rowmotion can be calculated by toggling at every element of the poset in order from top to bottom.
Let me demonstrate this on my usual example, by calculating the sequence of toggles; we will then see that the result is the same as rowmotion.



# Rowmotion for graphs

I want to define a version of rowmotion for independent sets of a directed graph $G$. An independent set of $G$ is a set of vertices no two elements of which are adjacent.

I will also need to assume that $G$ has no oriented cycles. (I will use the term acyclic for this — note that I only exclude oriented cycles.)

If $P$ is a poset, we can write $G(P)$ for the directed graph whose vertices are the elements of $P$, with an edge from $x$ to $y$ whenever $x > y$ in the poset. (Note that the directed graphs we get in this way are quite special: they satisfy transitivity.)

An independent set for $G(P)$ is nothing but an antichain for $P$. If this more general construction is applied to $G(P)$, we want to recover rowmotion for antichains in $P$ as defined previously.

# Tight orthogonal pairs of independent sets

Let $G$ be a directed, acyclic graph. This defines a poset on the vertices of $G$ by taking transitive closure (i.e., $x > y$ if there is a directed path from $x$ to $y$).

A pair of independent sets $(D, U)$ is called *orthogonal* if there is no arrow running from any $x \in D$ to any $y \in U$. (Note that this condition is not symmetric!)

A pair of orthogonal independent sets is called *tight* if whenever any element of $U$ is decreased, any element of $D$ is increased, or any element is added to $U$ or to $D$, the result is no longer an orthogonal pair.

We can see examples of these definitions in the following rather simple graph:

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

# Tight orthogonal pairs

**Proposition (T–Williams)**

*Given any independent set $D$, there is a unique independent set $I$ such that $(D, I)$ is a tight orthogonal pair.*

**Proof.**

$I$ can be constructed by reading up the graph (with respect to the partial order induced by the edges)) and greedily adding elements to $I$. As we consider each vertex of $G$, we add it provided there is no arrow to it from an element of $D$, and there is no arrow between it and any previously added element of $I$. □

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

Similarly, given an independent set $U$, there is a unique independent set $I$ such that $(I, U)$ is a tight orthogonal pair.

# Rowmotion for independent sets

Given an independent set $X$, we define $\rho(X)$ to be the unique independent set such that $(X, \rho(X))$ is a tight orthogonal pair.

Note that when we iterate this, we find $\rho(X)$ as the upper independent set corresponding to $X$, and then we view it as a lower independent set, and calculate the upper independent set corresponding to it, and continue.

If $P$ is a poset and $G = G(P)$, an independent set $X$ is an antichain, and the greedy algorithm for computing $\rho(X)$ will add the minimal elements of $P \setminus X^-$: i.e., it computes rowmotion for the poset $P$.

# Orbit sizes for rowmotion on a directed path

| Graph | Orbit sizes |
| --- | --- |
| $\bullet \longrightarrow \bullet$ | 3 |
| $\bullet \longrightarrow \bullet \longrightarrow \bullet$ | 2, 3 |
| $\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$ | 3, 5 |
| $\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$ | 2, 3, 8 |
| $\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$ | 3, 7, 11 |
| $\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$ | 2, 3, 5, 10, 14 |
| $\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$ | 2, 9, 13, 13, 17 |
| $\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$ | 2, 3, 8, 12, 12, 16, 16, 20 |

# Toggles

As I explained, rowmotion for antichains (or more precisely for order ideals) can be calculated as a composition of toggles. We will now see how to do something similar for independent sets.

Given a tight orthogonal pair $(D, U)$, and a vertex $v$, define the toggle at $v$ as follows:

- If $v$ is in neither $D$ nor $U$, return $(D, U)$ and stop.
- Otherwise, move $v$ into the other set.
- Remove the elements of $D$ that are less than $v$ and the elements of $U$ that are greater than $v$.
- Greedily add elements above $v$ to $U$ upwards and greedily add elements below $v$ to $D$ downwards.
- Return $(D, U)$.

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \xrightarrow{\tau_v} \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

# Rowmotion on independent sets via toggles

Recall the definition of toggle:

- If $v$ is in neither $D$ nor $U$, return $(D, U)$ and stop.
- Otherwise, move $v$ into the other set.
- Remove the elements of $D$ that are less than $v$ and the elements of $U$ that are greater than $v$.
- Greedily add elements above $v$ to $U$ upwards and greedily add elements below $v$ to $D$ downwards.
- Return $(D, U)$.

**Theorem (T–Williams)**

*Rowmotion on independent sets can be calculated by toggling at each vertex of G from top to bottom.*

Example (live action):

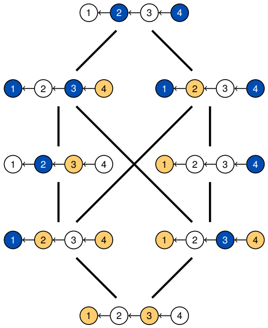
$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \;\rightarrow\; \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

# An order on independent sets

Define a relation on independent sets by $I' \gtrdot I$ if $(I, \rho(I))$ is obtained from $(I', \rho(I'))$ by a toggle at an element of $I'$.

Theorem (T–Williams)

*The relation $\gtrdot$ defines the cover relation of a poset.*




This poset is not necessarily a lattice, but if it is a lattice, it has some nice properties (left modular, extremal). All such lattices can be constructed in this way.

Thank you!