Mutually orthogonal cycle systems

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Joint work with:

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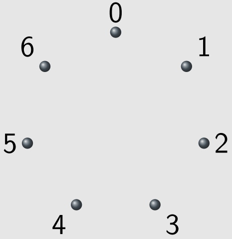
Graph decompositions

- Let Γ be a graph, and let $H_1, H_2, \ldots, H_t$ be subgraphs of Γ.
- The subgraphs $H_1, H_2, \ldots, H_t$ decompose Γ if their edge sets partition the edges of Γ.
- If $H_1 \simeq \cdots \simeq H_t \simeq H$, then we speak of an H-decomposition of Γ.

Example (A K_3-decomposition of K_7)



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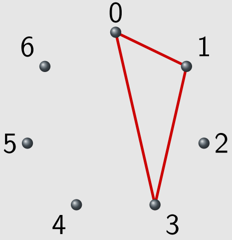

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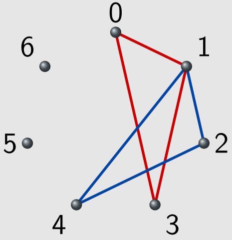

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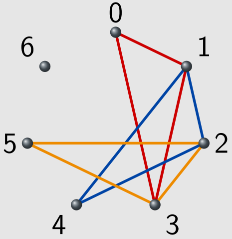

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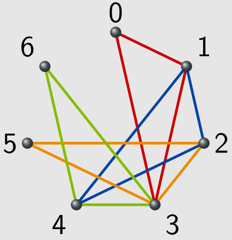

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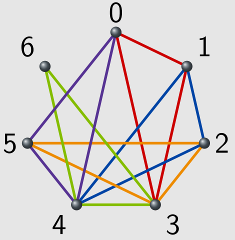

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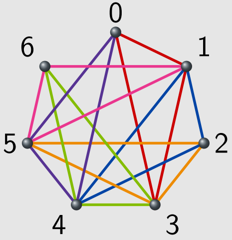

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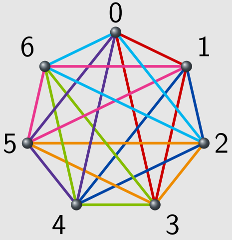

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Decompositions and Designs

Definition

A K_3-decomposition of K_n is called a Steiner triple system of order n, STS(n).

Theorem (Kirkman, 1847)

Let $n \in \mathbb{Z}^+$. There is an $STS(n)$ if and only if $n \equiv 1$ or 3 (mod 6).

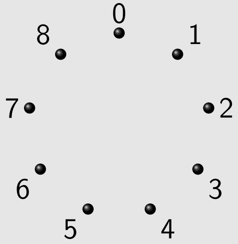
A K_k-decomposition of K_v is a:

- Balanced Incomplete Block Design BIBD$(v, k, 1)$
- Steiner system S$(2, k, v)$

Cycle systems

A C_ℓ-decomposition of K_n is called a ℓ-cycle system of order n.

Example (A 4-cycle system of order 9)



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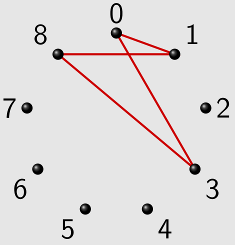

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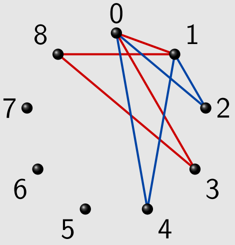

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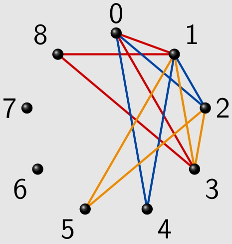

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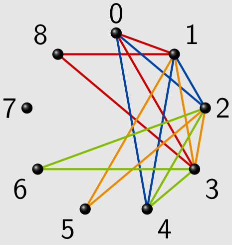

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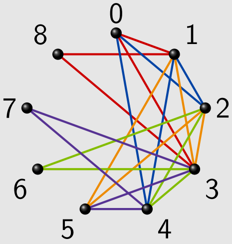

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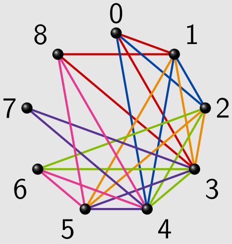

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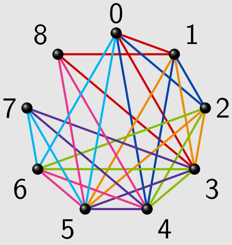

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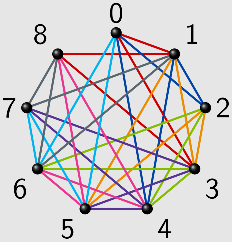

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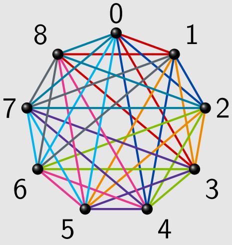

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Existence of ℓ-cycle systems

Theorem (Alspach, Gavlas, 2001; Šajna, 2002; see also Buratti 2003)

There exists an ℓ-cycle decomposition of K_n if and only if:

- *n is odd,*
- $3 \leq \ell \leq n$, *and*
- $\ell \mid \binom{n}{2}$

Given $\ell \geq 3$, we will refer to a value of n satisfying these conditions as ℓ-admissible.

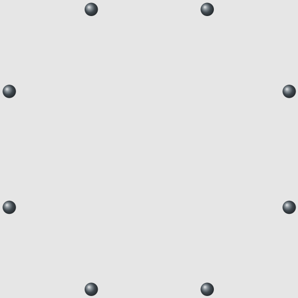
Cycle decompositions of the cocktail party graph

Theorem (Alspach, Gavlas, 2001; Šajna, 2002)

There exists an ℓ-cycle decomposition of $K_n - I$ if and only if:

- *n is even,*
- *$3 \leq \ell \leq n$, and*
- $\ell \mid \binom{n}{2} - n$

Example (A 4-cycle decomposition of $K_8 - I$)



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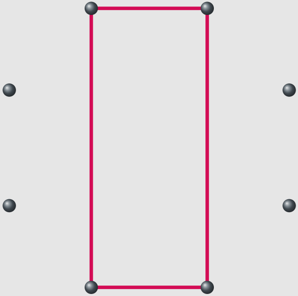

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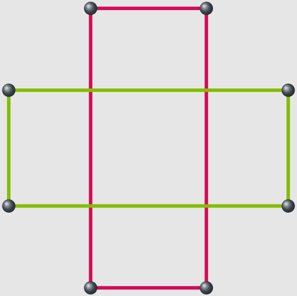

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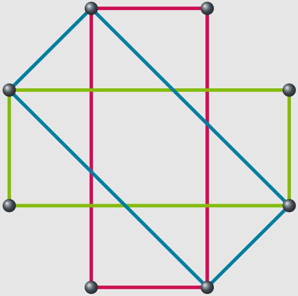

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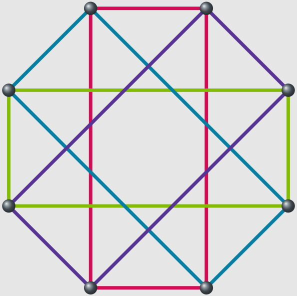

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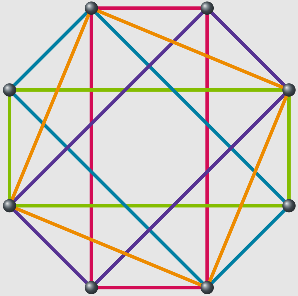

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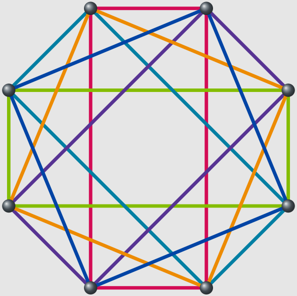

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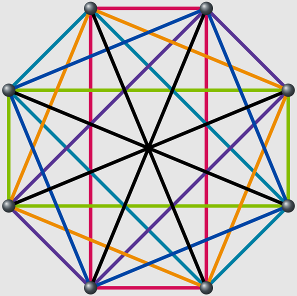

Cyclic cycle systems

A cycle system $\mathscr{C}$ is cyclic if it admits an automorphism which cyclically permutes the vertices.

In other words, we can take the vertex set to be $\mathbb{Z}_n$, and

$$(c_0, c_1, \ldots, c_{\ell-1}) \in \mathscr{C} \Rightarrow (c_0 + 1, c_1 + 1, \ldots, c_{\ell-1} + 1) \in \mathscr{C}.$$

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Necessary and sufficient conditions for existence of a cyclic C_ℓ-decomposition of K_n are known when:

- $n \equiv 1$ or $\ell \pmod{2\ell}$. (Buratti and Del Fra, 2003; Bryant, Gavlas and Ling, 2003; Buratti and Del Fra, 2004; Vietri, 2004)
- $\ell = n$ (Buratti and Del Fra, 2004)
- $\ell \leq 32$ (Wu and Fu, 2006)
- $\ell = 2p^\alpha$ or $3p$, p prime (Wu and Fu, 2006; Wu, 2013)
- ℓ even and $n > 2\ell$ (Wu, 2012)

Difference Families

Consider a cycle $C = (c_0, c_1, \ldots, c_{\ell-1})$ with vertices in $\mathbb{Z}_n$.

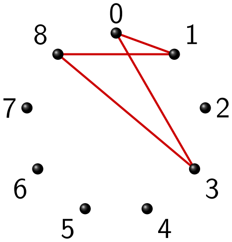
Its list of differences is the multiset $\Delta C = \{\pm(c_{i+1} - c_i) \mid 0 \leq i \leq \ell\}$ (where subscripts are taken modulo ℓ).

For a family $\mathcal{F}$ of cycles, $\Delta\mathcal{F}$ is the multiset union of the difference lists of its cycles.

If $\Delta\mathcal{F} = \mathbb{Z}_n \setminus \{0\}$, then $\mathcal{F}$ is a set of base cycles for a cyclic ℓ-cycle system of order n.

If such a family exists, then $n \equiv 1 \pmod{2\ell}$.

Example: A cyclic 4-cycle system of order 9



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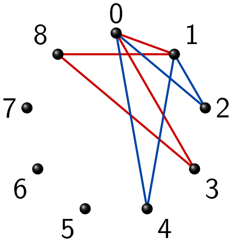

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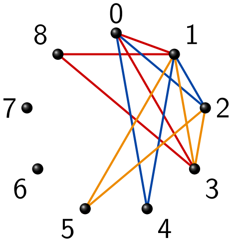

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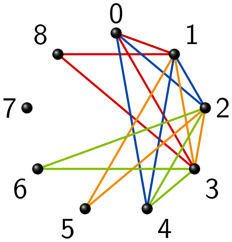

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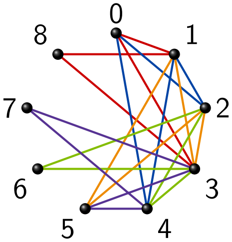

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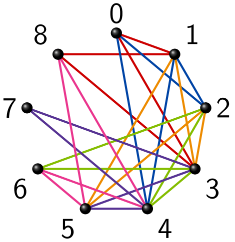

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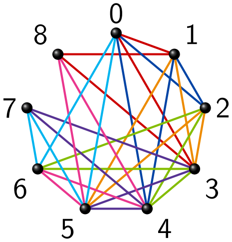

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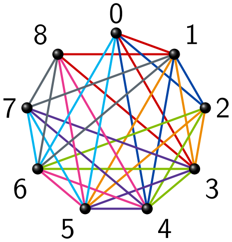

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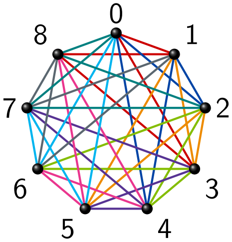
$(5, 6, 4, 8)$
$(6, 7, 5, 0)$
$(7, 8, 6, 1)$

Example: A cyclic 4-cycle system of order 9



$(0, 1, 8, 3)$
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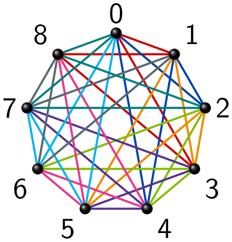

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We denote the orbit of the base cycle $(0, 1, 8, 3)$ under the action of $\mathbb{Z}_9$ by

$$[1, -2, 4, -3]_9.$$

Example

The cycles $(0, 1, 5, 3)$ and $(0, 5, 13, 7)$ are base cycles for a cyclic 4-cycle system of order 17.

Orbits: $[1, 4, -2, -3]_{17}$ and $[5, 8, -6, -7]_{17}$

Orthogonal cycle systems

Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be ℓ-cycle systems on the same vertex set.

These systems are orthogonal if any cycles $C \in \mathscr{C}_1$ and $C' \in \mathscr{C}_2$ share at most one edge.

Example

$\mathscr{B}$	$\mathscr{C}$	$\mathscr{D}$
$(0,1,8,3)$	$(0,1,2,3)$	$(0,1,7,6)$
$(1,2,0,4)$	$(0,2,5,4)$	$(0,2,3,5)$
$(2,3,1,5)$	$(0,5,1,6)$	$(0,3,8,7)$
$(3,4,2,6)$	$(0,7,2,8)$	$(0,4,2,8)$
$(4,5,3,7)$	$(1,3,6,4)$	$(1,2,7,4)$
$(5,6,4,8)$	$(2,4,7,6)$	$(1,3,4,5)$
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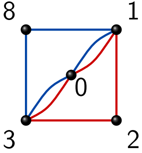

$\mathscr{B}$ and $\mathscr{C}$ are not orthogonal

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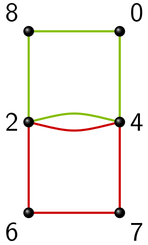

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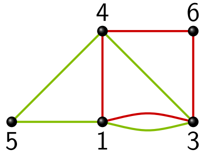

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Heffter arrays

Definition (Archdeacon, 2015)

A Heffter array $H(m, n; k, \ell)$ is an $m \times n$ array with entries from $\mathbb{Z}_{2mk+1} \setminus \{0\}$ such that:

- Each row contains k filled cells, and each column contains ℓ filled cells.
- Each row and column sums to $0 \pmod{2mk+1}$.
- For each $x \in \mathbb{Z}_{2mk+1} \setminus \{0\}$, exactly one of x and $-x$ appears as an entry.

If $m = n$ and $k = \ell$, we write $H(n; \ell)$.

Theorem (Archdeacon, Dinitz, Donovan, Yazıcı, 2015; Dinitz, Wanless, 2017; Cavenagh, Dinitz, Donovan, Yazıcı, 2019)

There is a square Heffter array $H(n;\ell)$ *if and only if* $3 \leq \ell \leq n$.

Example (An $H(8;7)$)

8	16		25	−27	−29	31	−24
−17	−6	23	−28	26	32	−30	
39	−10	−5	15		33	−35	−37
−38		−18	7	11	−36	34	40
−43	−45	47	−22	3	19		41
42	48	−46		−14	2	12	−44
	49	−51	−53	55	−21	1	20
9	−52	50	56	−54		−13	4

(Example taken from Costa, Morini, Pasotti and Pellegrini, 2018.)

Heffter arrays and orthogonal cycle systems

- Consider the entries of $H(n;\ell)$ as differences in $\mathbb{Z}_{2n\ell+1}$.

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- Similarly, if each column can be ordered appropriately, we get another cyclic ℓ-cycle system.
- These cycle systems are orthogonal.

Existence results for orthogonal cycle systems

Theorem (Costa, Morini, Pasotti, Pellegrini, 2018)

- *For $3 \leq \ell \leq 10$, there exists a pair of orthogonal cyclic k-cycle systems of order $2n\ell + 1$ whenever $n\ell \equiv 0$ or 3 (mod 4).*
- *Comparable result for the cocktail party graph.*

Theorem (Burrage, Donovan, Cavenagh, Yazıcı, 2020)

There is a pair of orthogonal cyclic ℓ-cycle systems of order $2n\ell + 1$ whenever

- $\ell \equiv 0$ *(mod 4)*
- $n \equiv 1$ *(mod 4) and* $\ell \equiv 3$ *(mod 4)*
- $n \equiv 0$ *(mod 4),* $\ell \equiv 3$ *(mod 4) and* $n \gg \ell$

Mutually orthogonal cycle systems

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- $\mu(\ell, n)$ denotes the maximum number of mutually orthogonal ℓ-cycle systems of order n
- $\mu'(\ell, n)$ denotes the maximum number of mutually orthogonal cyclic ℓ-cycle systems of order n

Upper Bounds

Lemma (AB, Cavenagh, Pike, 2022+)

- $\mu(\ell, n) \leq n - 2$
- $\mu(\ell, n) \leq \dfrac{(n-2)(n-3)}{2(\ell-3)}$
- *If* $2\ell^2 > n(n-1)$*, then* $\mu(\ell, n) \leq 1$.

  So if $\ell > \dfrac{n}{\sqrt{2}}$*, there is no pair of orthogonal cycle systems of order* n.
- $\mu'(\ell, n) \leq n - 3$

Mutually orthogonal 3-cycle systems

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Corollary

If $n \neq 7$ is 3-admissible, then $\mu(3, n) = n - 2$.

Asymptotic results

Theorem (Caro and Yuster, 2001)

Let H be a graph and $k \geq 1$ a fixed integer.

For any sufficiently large n such that K_n is H-decomposable, there exists a set of k pairwise orthogonal H-decompositions of K_n.

Corollary

For any sufficiently large ℓ-admissible n, there exists a set of k pairwise orthogonal ℓ-cycle systems of K_n.

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Lemma (AB, Cavenagh, Pike, 2022+)

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Theorem (AB, Cavenagh, Pike, 2022+)

If $n \equiv 1 \pmod 8$ *and* $n \geq 17$, *then* $\mu'(4,n) \geq \frac{n-1}{2}$.

Example: Order $n = 17$

- Take a 1-factorization of $K_4 = K_{(n-1)/4}$

$$F_1 = \{\{1,2\},\{3,4\}\} \quad F_2 = \{\{1,3\},\{2,4\}\} \quad F_3 = \{\{1,4\},\{2,3\}\}$$

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- From each edge $\{x, y\}$ of K_4 form a cycle $C_{x,y}$ with

$$\Delta(C_{x,y}) = \pm\{2x-1, 2x, 2y-1, 2y\}.$$

Each 1-factor yields base cycle for a cyclic 4-cycle system $\mathcal{F}_i$.

	Base cycles	Orbits
$\mathcal{F}_1$	$\{(0,1,5,3),(0,5,13,7)\}$	$\{[1,4,-2,-3]_{17},[5,8,-6,-7]_{17}\}$
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$\mathcal{F}_1'$	$\{(0,1,-2,-4),(0,5,-2,-8)\}$	$\{[1,-3,-2,4]_{17},[5,-7,-6,8]_{17}\}$
$\mathcal{F}_2'$	$\{(0,1,-4,-6),(0,3,-4,-8)\}$	$\{[1,-5,-2,6]_{17},[3,-7,-4,8]_{17}\}$
$\mathcal{F}_3'$	$\{(0,1,-6,-8),(0,3,-2,-6)\}$	$\{[1,-7,-2,8]_{17},[3,-5,-4,6]_{17}\}$

- Modify each $\mathcal{F}_i$ to get another system $\mathcal{F}_i'$.

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- Modify each $\mathcal{F}_i$ to get another system $\mathcal{F}_i'$.
- Replace $\mathcal{F}_1$ and $\mathcal{F}_1'$ with four specially constructed systems.

Even cycle systems

Theorem (AB, Cavenagh, Pike, 2022+)

Let $\ell \geq 4$ be even and $n \equiv 1 \pmod{2\ell}$. Then

$$\mu(\ell, n) = \Omega\left(\frac{n}{\ell^2}\right).$$

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Specifically,

Theorem (AB, Cavenagh, Pike, 2022+)

Let $\ell \geq 4$ be even and $n \equiv 1 \pmod{2\ell}$. Then

$$\mu'(\ell, n) \geq \frac{n-1}{2\ell(a\ell + b)} - 1,$$

where

$$(a, b) = \begin{cases} (4, -2), & \text{if } \ell \equiv 0 \pmod 4 \\ (24, -18), & \text{if } \ell \equiv 2 \pmod 4. \end{cases}$$

Balanced sets of differences

Definition

A set $D = \{d_1, d_2, \ldots, d_{2k}\}$ of positive integers with $d_1 < d_2 < \ldots < d_{2k}$ is balanced if there exists $t \in [1, k]$ such that

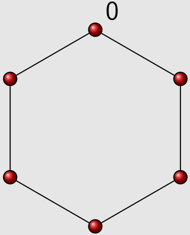
$$\sum_{i=1}^{2t} (-1)^i d_i = \sum_{i=2t+1}^{2k} (-1)^i d_i.$$

Lemma

If D is balanced, then there is a $2k$-cycle C with vertices in $[-d_{2k}, d_{2k-1}]$ such that $\Delta C = \pm D$.

Example (Forming cycles from balanced sets)

D: 1 2 3 4 6 8



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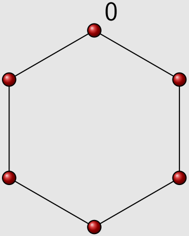

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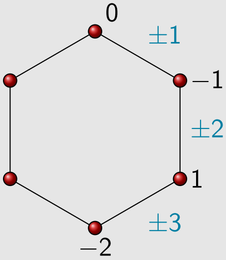

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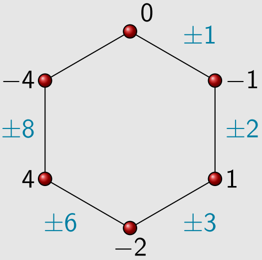

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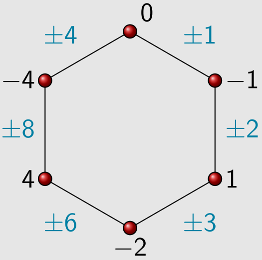

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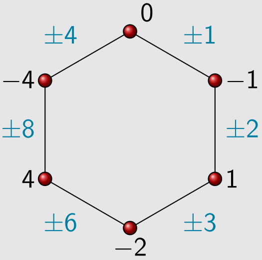

So if we can partition the set $\{1, 2, \ldots, (n-1)/2\}$ into balanced $2k$-sets, then there is a cyclic $2k$-cycle system of order n.

$4k$-cycle systems of order $n = 8kN + 1$

- For a pair (d, e) with $d + e = N$, we can form a cyclic $4k$-cycle system $\mathscr{C}_d$ of order $8kN + 1$.

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- For a pair (d, e) with $d + e = N$, we can form a cyclic $4k$-cycle system $\mathscr{C}_d$ of order $8kN + 1$.
- We form d balanced $4k$-sets that partition

  $$\{1, \ldots, 4kd\},$$

  and e balanced $4k$-sets that partition

  $$\{4kd + 1, \ldots, 4kN\}.$$

Example: A cyclic 12-cycle system of order $97 = 24 \cdot 4 + 1$

Let $d = 1$ and $e = 3$. We get the following balanced sets of differences.

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$$

$$13, 16, 19, 22, 25, 28, 31, 34, 37, 40, 43, 46$$
$$14, 17, 20, 23, 26, 29, 32, 35, 38, 41, 44, 47$$
$$15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48$$

These yield starter cycles

$$(0, -1, 1, -2, 2, -3, 4, -4, 5, -5, 6, -6)$$

$$(0, -13, 3, -16, 6, -19, 12, -22, 15, -25, 18, -28)$$
$$(0, -14, 3, -17, 6, -20, 12, -23, 15, -26, 18, -29)$$
$$(0, -16, 3, -18, 6, -21, 12, -24, 15, -27, 18, -30)$$

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Lemma

If d and d' are distinct integers with $\frac{N}{2} - \frac{N}{16k-2} < d, d' < \frac{N}{2}$, then for cycles $C \in \mathscr{C}_d$ and $C' \in \mathscr{C}_{d'}$, $\Delta(C) \cap \Delta(C') = \emptyset$ or $\{\pm t\}$.

Hence $\mathscr{C}_d$ and $\mathscr{C}_{d'}$ are orthogonal.

Computational results

For $n = 2\ell + 1$, we have the following computational results:

ℓ	n	$\mu'(\ell, n)$
3	7	2
4	9	2
5	11	4
6	13	5
7	15	8
8	17	8
9	19	≥ 8
10	21	≥ 8
11	23	≥ 8

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Question

Are there any ℓ-admissible values n with $\mu'(\ell, n) = n - 3$?

Future directions

- Find constructions of mutually orthogonal (cyclic) odd cycle systems.
- Find improved lower bounds on $\mu(\ell, n)$ when $n \equiv 1 \pmod{2\ell}$.
- Find lower bounds on $\mu(\ell, n)$ for other ℓ-admissible values n.
- Investigate mutually orthogonal (cyclic) cycle decompositions of $K_n - I$.

Thanks!

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