Counting loops and multi-edges in the superposition of a fixed graph and a random graph

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Motivation: Tree-Rooted Graphs

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A *tree-rooted graph* is a triple (G, T, γ) where G is a connected graph, T is a spanning tree of G, and $\gamma = uv$ is a distinct, oriented edge with $uv \in e(G) \setminus e(T)$. We view T as a rooted tree with root u.

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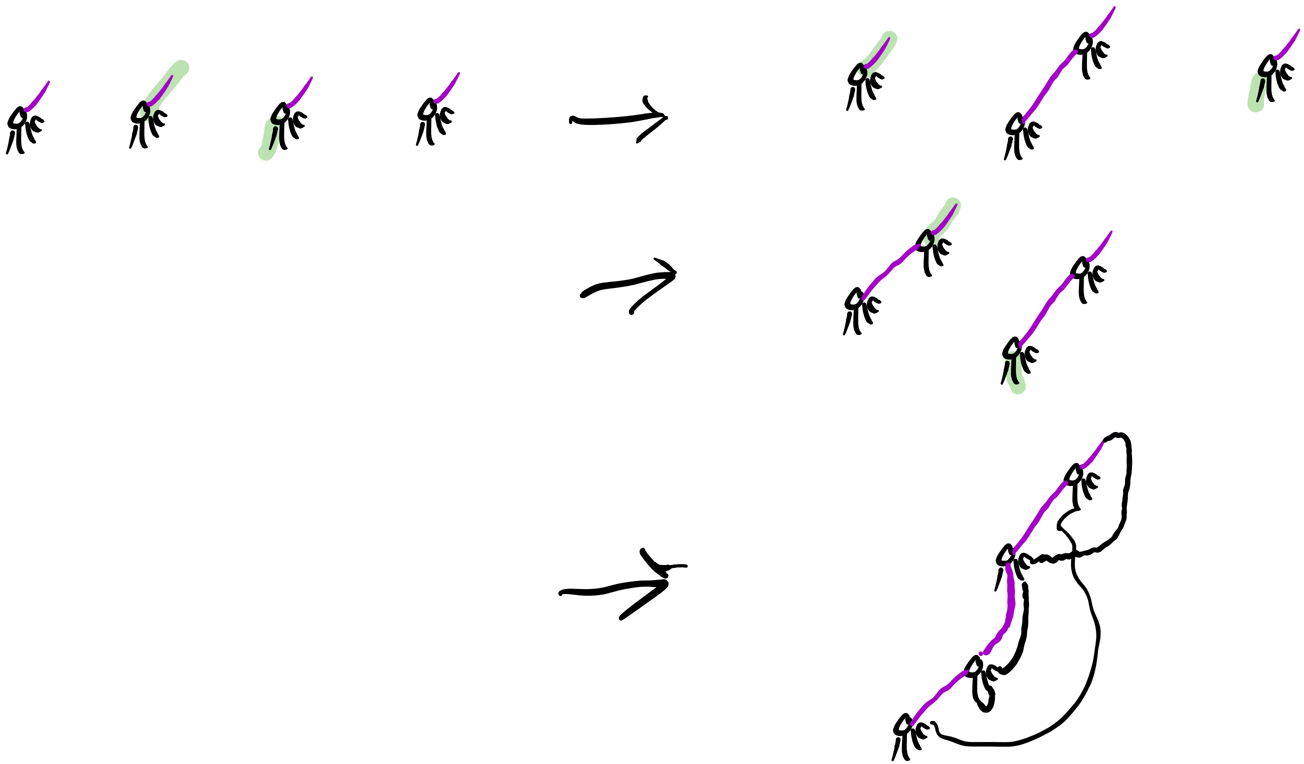
Theorem

For each $n \geq 1$, let $\mathrm{d}^n = (d^n(v)1 \leq v \leq n)$ be a degree sequence, and let (G_n, T_n, Γ_n) be chosen uniformly at random among all simple tree-rooted graphs with vertex set $[n]$ and degree sequence d^n. Then (under certain conditions on the limiting distribution of d^n) there exists a constant σ (dependent on the limiting distribution) such that

$$\frac{\sigma}{n^{1/2}} T_n \overset{\text{dist}}{\to} \mathcal{T}$$

as $n \to \infty$ with respect to the Gromov-Hausdorff-Prokhorov topology, where $\mathcal{T}$ is the Brownian continuum random tree.

Motivation: Constructing a Tree-Rooted Graph



Motivation: Problem/Solution

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Solution: Prove that, under this construction, the probability that the graph is simple asymptotically approaches a constant greater that zero.

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For each $n \geq 1$ let $\mathrm{d}^n = (d^n(v), 1 \leq v \leq n)$ be a degree sequence. Let $G_n = G(\mathrm{d}^n)$ be a random graph with degree sequence d^n, distributed according to the configuration model.

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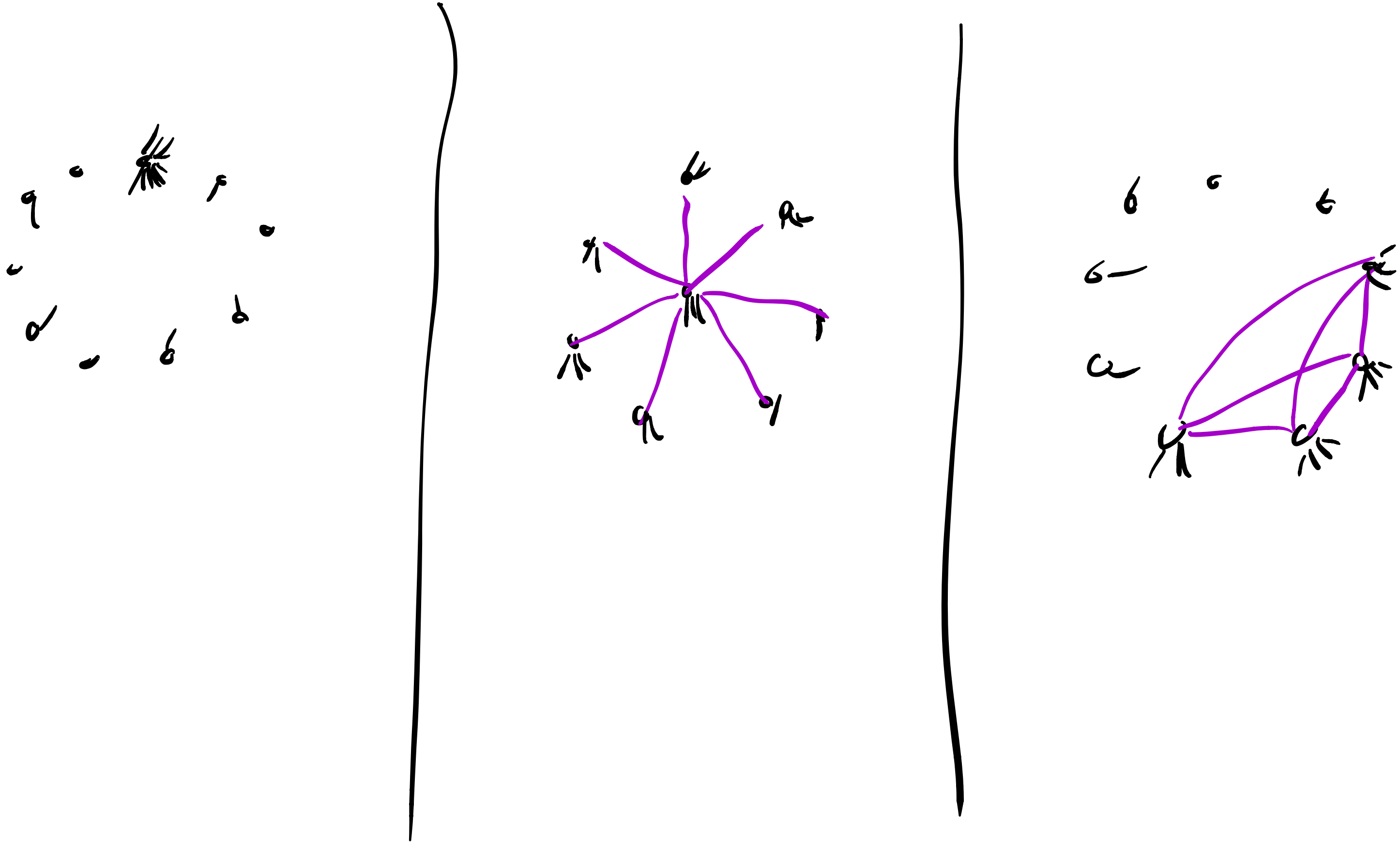
For each $n \geq$, let $\mathfrak{G}_n$ be the graph with $v(\mathfrak{G}_n) = [n]$ and $e(\mathfrak{G}_n) = e(H_n) \cup e(G_n)$. We are interested in the asymptotics of $\mathbf{P}\left\{\mathfrak{G}_n \text{ is a simple graph}\right\}$.

Main Result: Problematic Examples



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First, for $(d^n, n \geq 1)$, we require at least that $\max_{v\in[n]}\{d^n(v)\} = o(n^{1/2})$. In "The Probability That a Random Graph is Simple", Svante Janson shows that the correct condition for ensuring $\mathbf{P}\{G_n \text{ is a simple graph}\} > 0$ is a bounded second moment on the probability distribution of d^n.

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Second, for $(H_n, n \geq 1)$, we require $\max_{v \in [n]}\{\deg_{H_n}(v)\} = o(n)$.

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Second, for $(H_n, n \geq 1)$, we require $\max_{v\in[n]}\{\deg_{H_n}(v)\} = o(n)$.

Lastly, we require some sort of control on $\sum_{uv\in e(H^n)} d^n(u)d^n(v)$.

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Let $p^n = (p^n(k), k \geq 0)$ be the probability distribution of d^n. Furthermore, for any probability distribution $p = (p(k), k \geq 0)$, let $\mu_1(p) = \sum_{k \geq 0} kp(k)$ and $\mu_2(p) = \sum_{k \geq 0} k^2 p(k)$.

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Condition $*$*:* There exists a probability distribution $p = (p(k), k \geq 0)$ with $\mu_2(p) \in [0, \infty)$ and $p(0) < 1$ such that the following holds:
First, $p^n \to p$ pointwise and $\mu_2(p^n) \to \mu_2(p)$. Second, there are non-negative numbers $(\alpha(a, b), a, b \geq 0)$ such that for any $a, b \geq 0$

$$\alpha^n(a, b) := \frac{1}{n} |\{uv \in e(H_n) : d^n(u) = a, d^n(v) = b\}| \to \alpha(a, b),$$

and

$$\sum_{k,\ell\geq 0} kl\alpha^n(k, \ell) \to \sum_{k,\ell\geq 0} kl\alpha(k, \ell) < \infty \tag{1}$$

Main Result: Theorem

Let $(H_n, d^n, p^n, n \geq 1)$ and p be given, and let G_n be a random graph with degree sequence d^n, distributed according to the configuration model. Suppose that Condition $*$ holds, and that $\max_{v \in [n]}\{\deg_{H_n}(v)\} = o(n)$. Let L_n be the number of vertices G_n that contain loops, M_n be the number of edges in $e(G_n) \setminus e(H_n)$ with multiplicity greater than 1, and N_n be the number of edges H_n that are also edges in G_n.

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Theorem

Let $\nu = (\mu_2(p)/\mu_1(p)) - 1$ *and* $\eta = \frac{1}{\mu_1(p)} \sum_{i,j \geq 1} ij\alpha(i,j)$.

$$\|\mathrm{Dist}(L_n, M_n, N_n) - \mathrm{Poi}(\nu/2) \otimes \mathrm{Poi}(\nu^2/4) \otimes \mathrm{Poi}(\eta)\|_{\mathrm{TV}} \to 0$$

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An immediate corollary to this theorem is:

$$\mathbf{P}\{\mathfrak{G}_n \text{ is simple}\} \to \exp(-\nu/2 - \nu^2/4 - \eta)$$

as $n \to \infty$.

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For this section we fix n and simplify our notation like so:

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Theorem (Theorem 2.6 in Hofstad)

A vector of integer-valued random variables $\left((X_{1,n},\ldots,X_{d,n})\right)_{n\geq 1}$ *converges in distribution to a vector of independent Poisson random variables with parameters* $\lambda_1,\ldots,\lambda_d$ *when, for all* $r_1,\ldots,r_d \in \mathbb{N}$,

$$\lim_{n\to\infty} \mathbf{E}\left((X_{1,n})_{r_1}\ldots(X_{d,n})_{r_d}\right) = \lambda_1^{r_1}\ldots\lambda_d^{r_d}.$$

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The goal of this section is to find a deterministic bound for $\mathbf{E}\left((L)_q(M)_r(N)_s\right)$.

Deterministic Bounds: Set up

For vertices $u, v \in [n]$ and $i \in [d(u)], j \in [d(v)]$, let $\mathbf{1}_{[ui,vj]}$ be the indicator of the event that half-edge ui is matched with half-edge vj in G.

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$$\begin{aligned}
\mathcal{L} &= \mathcal{L}(G) = \{(ui, uj) : u \in [n], i, j \in [d(u)], i < j\}, \\
\mathcal{M} &= \mathcal{M}(G, H) = \{((ui_1, vj_1), (ui_2, vj_2)) : u, v \in [n], uv \notin e(H) \\
&\qquad i_1, i_2 \in [d(u)], j_1, j_2 \in [d(v)], u < v, i_1 < i_2, j_1 \neq j_2\}, \\
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\end{aligned}$$

and let us re-define L, M and N as

$$\begin{aligned}
L &= L(G) = \sum_{(ui,uj)\in\mathcal{L}} \mathbf{1}_{[ui,uj]}\ , \\
M &= M(G, H) = \sum_{((ui_1,vj_1),(ui_2 vj_2))\in\mathcal{M}} \mathbf{1}_{[(ui_1,vj_1)]}\mathbf{1}_{[(ui_2 vj_2)]}\ , \text{ and} \\
N &= N(G, H) \sum_{(ui,vj)\in\mathcal{N}} \mathbf{1}_{[ui,vj]}\ .
\end{aligned}$$

Deterministic Bounds: Proposition

Let $d_{\max} = \max_{i \in [n]}\{d(i)\}$.

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Proposition

For any positive integers $q, r, s \in \mathbb{N}$,

$$\left| \mathbf{E}\left((L)_q(M)_r(N)_s\right) - \frac{(|\mathcal{L}|)_q(|\mathcal{M}|)_r(|\mathcal{N}|)_s}{\prod_{i=0}^{q+2r+s-1} 2m-1-2i} \right| \leq C(S_1 + S_2)$$

where $C = C(q, r, s)$, S_1 is defined by

$$S_1 = (|\mathcal{L}|)_q(|\mathcal{N}|)_s \sum_{k=1}^{r-1} (|\mathcal{M}|)_{r-k} \sum_{\ell=0}^{k} d_{\max}^{2\ell} \prod_{i=0}^{q+2r+s-1-(2k-\ell)} \frac{1}{2m-1-2i},$$

and S_2 is defined by the following identity:

$$
\begin{aligned}
S_2 \prod_{i=0}^{q+2r+s-1} (2m-1-2i) =& (|\mathcal{L}|)_{q-2}(|\mathcal{M}|)_r(|\mathcal{N}|)_s \sum_{1\le u\le n} d(u)^3 \\
&+ (|\mathcal{L}|)_{q-1}(|\mathcal{M}|)_{r-1}(|\mathcal{N}|)_s \sum_{1\le u\ne v\le n} d(u)^3 d(v)^2 \\
&+ (|\mathcal{L}|)_{q-1}(|\mathcal{M}|)_r(|\mathcal{N}|)_{s-1} \sum_{uv\in e(H)} d(u)^2 d(v) \\
&+ (|\mathcal{L}|)_q(|\mathcal{M}|)_{r-2}(|\mathcal{N}|)_s \sum_{\substack{1\le u\le n\\ u\notin\{v_1,v_2\}}} d(u)^3 d(v_1)^2 d(v_2)^2 \\
&+ (|\mathcal{L}|)_q(|\mathcal{M}|)_{r-1}(|\mathcal{N}|)_{s-1} \sum_{\substack{1\le u,v_1,v_2\le n\\ u,v_1,v_2 \text{distinct}\\ uv_2\in e(H)}} d(u)^2 d(v_1)^2 d(v_2) \\
&+ (|\mathcal{L}|)_q(|\mathcal{M}|)_r(|\mathcal{N}|)_{s-2} \sum_{uv_1,uv_2\in e(H)} d(u)d(v_1)d(v_2).
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$$\left| \mathbf{E}\left((N)_s\right) - \frac{(|\mathcal{N}|)_s}{\prod_{i=0}^{s-1} 2m-1-2i} \right| \leq CS$$

where $C = C(s)$ and S is defined by

$$S = \frac{(|\mathcal{N}|)_{s-2}}{\prod_{i=0}^{s-1} 2m-1-2i} \sum_{uv_1, uv_2 \in e(H)} d(u)d(v_1)d(v_2)$$

Deterministic Bounds: Proof

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Using this notation, we can say that

$$\mathbf{E}\left((N)_s\right) = \sum\nolimits^{\star} \mathbf{P}\left\{\mathbf{1}_{[z]} = 1\right\}. \tag{2}$$

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$$\mathbf{P}\left\{\mathbf{1}_{[z]} = 1\right\} = \prod_{i=0}^{s-1} \frac{1}{2m-1-2i}. \tag{3}$$

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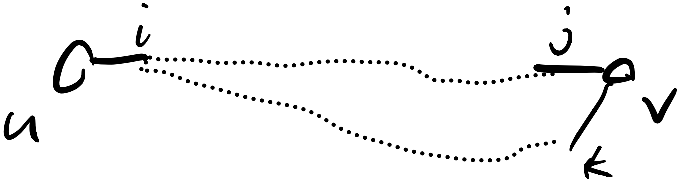
By the triangle inequality this implies

$$\left|\mathbf{E}\left((N)_s\right) - \frac{(|\mathcal{N}|)_s}{\prod_{i=0}^{s-1} 2m-1-2i}\right| \leq \sum_{z \text{ is conflicting}}^{\star} \left|er(z)\right|. \tag{4}$$

Deterministic Bounds: Proof

If z is conflicting, then there must be some $z_a, z_b \in z$ with $z_a = (ui, vj)$ and $z_b = (ui, vk)$, or with $z_a = (ui, vk)$ and $z_b = (uj, vk)$. Here is a visual representation of a conflicting z:

$z = (z_1, z_2, \ldots, z_s)$



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$$|\mathcal{C}| \leq C(|\mathcal{N}|)_{s-2} \sum_{uv_1, uv_2 \in e(H)} d(u)d(v_1)d(v_2)\,, \text{ and}$$

$$|er(z)| = \prod_{i=0}^{s-1} \frac{1}{2m-1-2i} \text{ for all } z \in \mathcal{C}.$$

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The second claim is obvious since $\mathbf{P}\left\{\mathbf{1}_{[z]} = 1\right\} = 0$ if $z \in \mathcal{C}$. The first claim requires a bit more work.

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Second, choose $(z_1, \dots, z_{a-1}, z_{a+1}, \dots, z_{b-1}, z_{b+1}, \dots, z_s)$ arbitrarily. The number of such choices is $(|\mathcal{N}|)_{s-2}$.

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Second, choose $(z_1, \ldots, z_{a-1}, z_{a+1}, \ldots, z_{b-1}, z_{b+1}, \ldots, z_s)$ arbitrarily. The number of such choices is $(|\mathcal{N}|)_{s-2}$.

Lastly, choose $uv_1, uv_2 \in e(H)$, $i \in [d(u)]$, $j \in [d(v_1)]$, and $k \in [d(v_2)]$. Then let $z_a = (ui, vj)$ and $z_b = (ui, vk)$. The number of choices here is bounded by

$$\sum_{uv_1, uv_2 \in e(H)} d(u)d(v_1)d(v_2).$$

Deterministic Bounds: Proof

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Putting this all together gives us

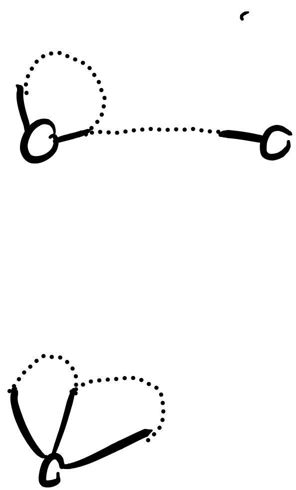
$$|\mathcal{C}| \le C(|\mathcal{N}|)_{s-2} \sum_{uv_1, uv_2 \in e(H)} d(u)d(v_1)d(v_2)$$

Deterministic Bounds: Proof

Finally, we have

$$\left| \mathbf{E}\left((N)_s\right) - \frac{(|\mathcal{N}|)_s}{\prod_{i=0}^{s-1} 2m-1-2i} \right| \leq \sum_{z\in\mathcal{C}}^{\star} |er(z)|$$
$$\leq \frac{(|\mathcal{N}|)_{s-2}}{\prod_{i=0}^{s-1} 2m-1-2i} \sum_{uv_1, uv_2 \in e(H)} d(u)d(v_1)d(v_2).$$

Deterministic Bounds: Other Types of Conflicts



Deterministic Bounds: Other Types of Conflicts

$$
\begin{aligned}
S_2 \prod_{i=0}^{q+2r+s-1} (2m-1-2i) = &(|\mathcal{L}|)_{q-2}(|\mathcal{M}|)_r(|\mathcal{N}|)_s \sum_{1\le u\le n} d(u)^3 \\
&+ (|\mathcal{L}|)_{q-1}(|\mathcal{M}|)_{r-1}(|\mathcal{N}|)_s \sum_{1\le u\ne v\le n} d(u)^3 d(v)^2 \\
&+ (|\mathcal{L}|)_{q-1}(|\mathcal{M}|)_{r}(|\mathcal{N}|)_{s-1} \sum_{uv\in e(H)} d(u)^2 d(v) \\
&+ (|\mathcal{L}|)_{q}(|\mathcal{M}|)_{r-2}(|\mathcal{N}|)_{s} \sum_{\substack{1\le u\le n \\ u\notin\{v_1,v_2\}}} d(u)^3 d(v_1)^2 d(v_2)^2 \\
&+ (|\mathcal{L}|)_{q}(|\mathcal{M}|)_{r-1}(|\mathcal{N}|)_{s-1} \sum_{\substack{1\le u,v_1,v_2\le n \\ u,v_1,v_2 \text{distinct} \\ uv_2\in e(H)}} d(u)^2 d(v_1)^2 d(v_2) \\
&+ (|\mathcal{L}|)_{q}(|\mathcal{M}|)_{r}(|\mathcal{N}|)_{s-2} \sum_{uv_1,uv_2\in e(H)} d(u)d(v_1)d(v_2).
\end{aligned}
$$

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$$S_1 = (|\mathcal{L}|)_q (|\mathcal{N}|)_s \sum_{k=1}^{r-1} (|\mathcal{M}|)_{r-k} \sum_{\ell=0}^{k} d_{\max}^{2\ell} \prod_{i=0}^{q+2r+s-1-(2k-\ell)} \frac{1}{2m-1-2i},$$

Asymptotic Results: Lemma

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Lemma

$$\sum_{v \in [n]} d(v) = O(n),$$

$$\sum_{v \in [n]} (d(v))^2 = O(n),$$

$$\sum_{uv \in e(H)} d(u)d(v) = O(n), \quad \textit{and}$$

$$\sup_{u \in [n]} \sum_{v : uv \in e(H)} d(v) = o(n).$$

Asymptotic Results: Proof (time dependent)

Tying it Together: Deterministic to Asymptotic

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We want

$$\mathbf{E}\left((N)_s\right) = (1 + o(1)) \frac{(|\mathcal{N}|)_s}{\prod_{i=0}^{s-1} 2m - 1 - 2i}$$

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So we need to show

$$\sum_{uv_1, uv_2 \in e(H)} d(u)d(v_1)d(v_2) = o\left((|\mathcal{N}|)^2\right)$$

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Assume $|\mathcal{N}| = \Theta(n)$ (otherwise $\mathbf{E}(N) \to 0$).

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$$\sum_{\substack{uv_1, uv_2 \in e(H) \\ v_1 \neq v_2}} d(u)d(v_1)d(v_2) = \sum_{uv \in e(H)} d(u)d(v) \cdot \sum_{w : uw \in e(H)} d(w)$$

$$\leq \sum_{uv \in e(H)} d(u)d(v) \cdot \left(\sup_{u \in [n]} \sum_{w : uw \in e(H)} d(w) \right)$$

$$= |\mathcal{N}| \cdot o(n)$$

$$= o(|\mathcal{N}|^2).$$

Tying it Together: Last Step

$$\begin{aligned}\frac{(|\mathcal{N}|)_s}{\prod_{i=0}^{s-1} 2m-1-2i} &= (1+o(1))\frac{|\mathcal{N}|^s}{(2m)^s} \\ &= (1+o(1))\left(\frac{\sum_{uv\in e(H)} d(u)d(v)}{\sum_{u\in[n]} d(u)}\right) \\ &= (1+o(1))\left(\frac{\sum_{i,j\geq 1} ij\alpha^n(i,j)}{\mu_1(p^n)}\right) \\ &= (1+o(1))\eta^s.\end{aligned}$$

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$$= (1+o(1))\left(\frac{\sum_{i,j\geq 1} ij\alpha^n(i,j)}{\mu_1(p^n)}\right)$$
$$= (1+o(1))\eta^s.$$

Hence,

$$\mathbf{E}\left((N)_s\right) \to \eta^s.$$

Further Questions

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$\frac{1}{n}\sum_{uv\in e(H)} d(u)d(v)$ *converges.*

The max degree in H is $o(n)$.

Can we relax any of these constraints?

Thank you!

Our paper: "Random Tree-Weighted Graphs" (arXiv:2008.12167)