The Orthogonal Colouring Game

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supported by AARMS

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Atlantic Graph Theory Seminar - 2022

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Outline

Orthogonal Latin Square Colouring Game: Ruleset

Two players: Alice and Bob

- Board: a pair of $n \times n$ empty grids. Alice owns the first grid, Bob owns the second grid.

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- Moves: Alternate turns. Fill one cell of either grid with an integer $1, \ldots, m$.
- Conditions:
  - Latin property: no repeated integers in a row or column.
  - Orthogonality: ordered pairs appear at most once in the superimposed grids.

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Two players: Alice and Bob

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- Conditions:
  - Latin property: no repeated integers in a row or column.
  - Orthogonality: ordered pairs appear at most once in the superimposed grids.

How to win: # entries in players' grid is their final score. Same score: Draw. Otherwise, higher score wins.

How to Play: Example

Suppose $m = 3$.

1		

Owned by Alice

Owned by Bob

Pairs:

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Owned by Alice

	1	

Owned by Bob

Pairs:

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1		
	1	

Owned by Alice

	1	

Owned by Bob

Pairs:

How to Play: Example

Suppose $m = 3$.

1		
	1	

Owned by Alice

	1	
		2

Owned by Bob

Pairs:

How to Play: Example

Suppose $m = 3$.

1		
	1	
		1

Owned by Alice

	1	
		2

Owned by Bob

Pairs:

How to Play: Example

Suppose $m = 3$.

1		
	1	
		1

Owned by Alice

3	1	x
	x	2

Owned by Bob

Pairs: (1,3)

How to Play: Example

Suppose $m = 3$.

1		
	1	
2		1

Owned by Alice

3	1	x
	x	2

Owned by Bob

Pairs: (1,3)

How to Play: Example

Suppose $m = 3$.

1		
	1	
2		1

Owned by Alice

3	1	x
	x	2
		1

Owned by Bob

Pairs: (1,3) (1,1)

How to Play: Example

Suppose $m = 3$.

1		
	1	2
2		1

Owned by Alice

3	1	x
	x	2
x		1

Owned by Bob

Pairs: (1,3) (1,1) (2,2)

How to Play: Example

Suppose $m = 3$.

1		
	1	2
2		1

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	x	2
x	3	1

Owned by Bob

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	1	2
2	3	1

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3	1	x
	x	2
x	3	1

Owned by Bob

Pairs: (1,3) (1,1) (2,2) (3,3)

How to Play: Example

Suppose $m = 3$.

1		
	1	**2**
2	**3**	**1**

Owned by Alice

3	**1**	X
1	X	**2**
X	**3**	**1**

Owned by Bob

Pairs: (1,3) (1,1) (2,2) (3,3)

How to Play: Example

Suppose $m = 3$.

1	2	
	1	2
2	3	1

Owned by Alice

3	1	x
1	x	2
x	3	1

Owned by Bob

Pairs: (1,3) (1,1) (2,2) (3,3) (2,1)

How to Play: Example

Suppose $m = 3$.

1	2	3
	1	2
2	3	1

Owned by Alice

3	1	x
1	x	2
x	3	1

Owned by Bob

Pairs: (1,3) (1,1) (2,2) (3,3) (2,1)

How to Play: Example

Suppose $m = 3$.

1	**2**	**3**
3	**1**	**2**
2	**3**	**1**

Owned by Alice

3	**1**	x
1	x	**2**
x	**3**	**1**

Owned by Bob

Pairs: (1,3) (1,1) (2,2) (3,3) (2,1) (3,1)

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3	**1**	**2**
2	**3**	**1**

Owned by Alice

3	**1**	x
1	x	**2**
x	**3**	**1**

Owned by Bob

Pairs: (1,3) (1,1) (2,2) (3,3) (2,1) (3,1)

Alice's score: 9, Bob's score: 6. ALICE WINS!

Can Bob do better?

Can Bob do better?

1		

Owned by Alice

Owned by Bob

Pairs:

Can Bob do better?

1		

Owned by Alice

		1

Owned by Bob

Pairs:

Can Bob do better?

1		
	1	

Owned by Alice

		1

Owned by Bob

Pairs:

Can Bob do better?

1		
	1	

Owned by Alice

		1
	1	

Owned by Bob

Pairs: (1,1)

Can Bob do better?

1		
	1	
		1

Owned by Alice

		1
	1	

Owned by Bob

Pairs: (1,1)

Can Bob do better?

1		
	1	
		1

Owned by Alice

		1
	1	
1		

Owned by Bob

Pairs: (1,1)

Can Bob do better?

1		
	1	
2		1

Owned by Alice

		1
	1	
1		

Owned by Bob

Pairs: (1,1) (2,1)

Can Bob do better?

1		
	1	
2		1

Owned by Alice

		1
	1	
1		2

Owned by Bob

Pairs: (1,1) (2,1) (1,2)

Can Bob do better?

1		
	1	2
2		1

Owned by Alice

		1
	1	
1		2

Owned by Bob

Pairs: (1,1) (2,1) (1,2)

Can Bob do better?

1		
	1	2
2		1

Owned by Alice

		1
2	1	
1		2

Owned by Bob

Pairs: (1,1) (2,1) (1,2)

Can Bob do better?

1		
	1	2
2	3	1

Owned by Alice

		1
2	1	
1		2

Owned by Bob

Pairs: (1,1) (2,1) (1,2)

Can Bob do better?

1		
	1	2
2	3	1

Owned by Alice

		1
2	1	
1	3	2

Owned by Bob

Pairs: (1,1) (2,1) (1,2) (3,3)

Can Bob do better?

1	2	
	1	2
2	3	1

Owned by Alice

		1
2	1	
1	3	2

Owned by Bob

Pairs: (1,1) (2,1) (1,2) (3,3)

Can Bob do better?

1	2	
	1	2
2	3	1

Owned by Alice

	2	1
2	1	
1	3	2

Owned by Bob

Pairs: (1,1) (2,1) (1,2) (3,3) (2,2)

Can Bob do better?

1	2	
3	1	2
2	3	1

Owned by Alice

	2	1
2	1	
1	3	2

Owned by Bob

Pairs: (1,1) (2,1) (1,2) (3,3) (2,2) (3,2)

Can Bob do better?

1	2	
3	1	2
2	3	1

	2	1
2	1	3
1	3	2

Owned by Alice Owned by Bob

Pairs: (1,1) (2,1) (1,2) (3,3) (2,2) (3,2) (2,3)

Can Bob do better?

1	2	3
3	1	2
2	3	1

Owned by Alice

	2	1
2	1	3
1	3	2

Owned by Bob

Pairs: (1,1) (2,1) (1,2) (3,3) (2,2) (3,2) (2,3) (3,1)

Can Bob do better?

1	2	3
3	1	2
2	3	1

Owned by Alice

3	2	1
2	1	3
1	3	2

Owned by Bob

Pairs: (1,1) (2,1) (1,2) (3,3) (2,2) (3,2) (2,3) (3,1) (1,3)

Can Bob do better?

1	2	3
3	1	2
2	3	1

Owned by Alice

3	2	1
2	1	3
1	3	2

Owned by Bob

Pairs: (1,1) (2,1) (1,2) (3,3) (2,2) (3,2) (2,3) (3,1) (1,3)

Alice's score: 9, Bob's score: 9. DRAW!

What if Alice chooses a cell in Bob's square?

Changing Alice's Strategy

1		

Owned by Alice

Owned by Bob

Pairs:

Changing Alice's Strategy

1		

Owned by Alice

		1

Owned by Bob

Pairs:

Changing Alice's Strategy

1	2	

Owned by Alice

		1

Owned by Bob

Pairs:

Changing Alice's Strategy

1	2	

Owned by Alice

	2	1

Owned by Bob

Pairs: (2,2)

Changing Alice's Strategy

1	2	
	1	

Owned by Alice

	2	1

Owned by Bob

Pairs: (2,2)

Changing Alice's Strategy

1	2	
	1	

Owned by Alice

	2	1
	1	

Owned by Bob

Pairs: (2,2) (1,1)

Changing Alice's Strategy

1	2	
	1	

Owned by Alice

x	2	1
3	1	

Owned by Bob

Pairs: (2,2) (1,1)

Changing Alice's Strategy

1	2	x
	1	3

Owned by Alice

x	2	1
3	1	

Owned by Bob

Pairs: (2,2) (1,1)

Changing Alice's Strategy

1	2	x
	1	3
	3	

Owned by Alice

x	2	1
3	1	

Owned by Bob

Pairs: (2,2) (1,1)

Changing Alice's Strategy

1	2	x
	1	3
	3	

Owned by Alice

x	2	1
3	1	
	3	

Owned by Bob

Pairs: (2,2) (1,1) (3,3)

Changing Alice's Strategy

1	2	x
	1	3
	3	

x	2	1
3	1	x
	3	2

Owned by Alice

Owned by Bob

Pairs: (2,2) (1,1) (3,3)

Changing Alice's Strategy

1	2	x
x	1	3
2	3	

Owned by Alice

x	2	1
3	1	x
	3	2

Owned by Bob

Pairs: (2,2) (1,1) (3,3)

Changing Alice's Strategy

1	2	x
x	1	3
2	3	1

Owned by Alice

x	2	1
3	1	x
	3	2

Owned by Bob

Pairs: (2,2) (1,1) (3,3) (1,2)

Changing Alice's Strategy

1	2	x
x	1	3
2	3	1

Owned by Alice

x	2	1
3	1	x
1	3	2

Owned by Bob

Pairs: (2,2) (1,1) (3,3) (1,2) (2,1)

Changing Alice's Strategy

1	**2**	x
x	**1**	**3**
2	**3**	**1**

Owned by Alice

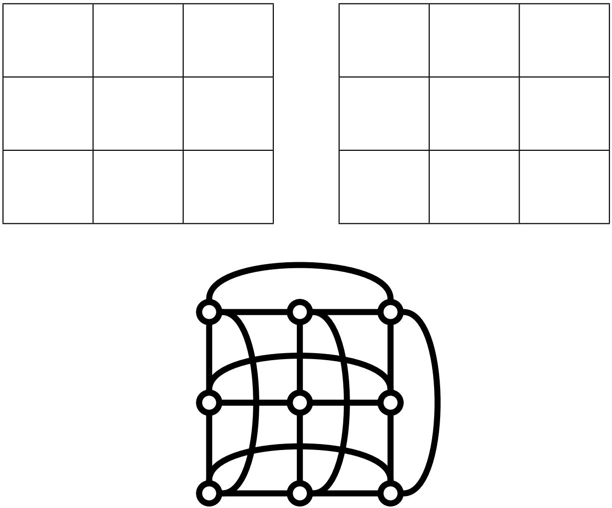
x	**2**	**1**
3	**1**	x
1	**3**	**2**

Owned by Bob

Pairs: (2,2) (1,1) (3,3) (1,2) (2,1)

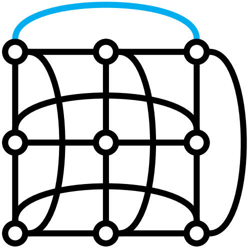
Alice's score: 7, Bob's score: 7. DRAW!

What is Bob doing?



What is Bob doing?

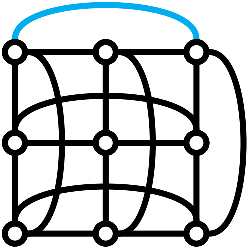
1		



What is Bob doing?

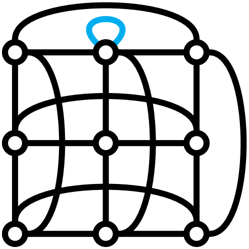
1		

		1



What is Bob doing?

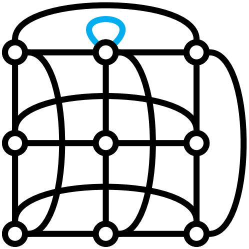
	2	



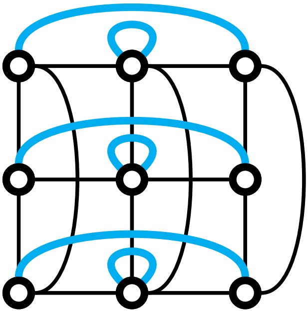
What is Bob doing?

	2	

	2	



What is Bob doing?



Generalizing the Game: Definition Interlude

Let G be a graph, $u, v \in V(G)$, and $\{1, \ldots, m\}$ a set of colours.

- $c_i(y)$: colour assigned $y \in V(G)$ in colouring i.

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Proper Colouring: Adjacent vertices receive different colours.

Orthogonal Colouring: Let i and j be a pair of orthogonal colourings of a graph G. We then have that

$$\text{if } c_i(u) = c_i(v), \text{ then } c_j(u) \neq c_j(v).$$

Example

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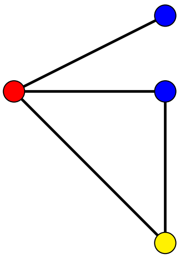
Example: $m = 3$



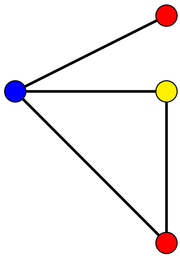
Pairs

(Blue, Red)

(Red, Blue)

(Blue, Yellow)

(Yellow, Red)



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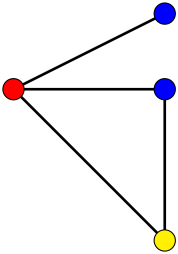


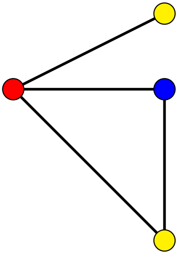
Pairs

(Red, Red)

(Blue, Blue)

(Blue, Yellow)

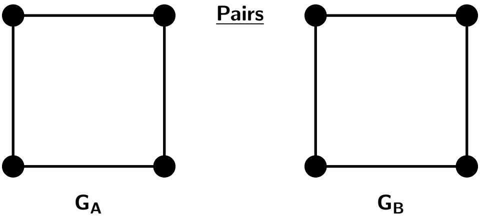
(Yellow, Yellow)



THE ORTHOGONAL COLOURING GAME: Ruleset

- Board: Disjoint isomorphic copies of a finite graph G, G_A and G_B. Alice owns G_A. Bob owns G_B.
- Moves: Colour a vertex in either graph from $\{1, \ldots, m\}$, satisfying (1) proper colouring; (2) orthogonality.

Example: $m = 2$



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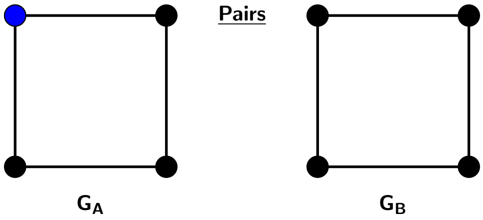

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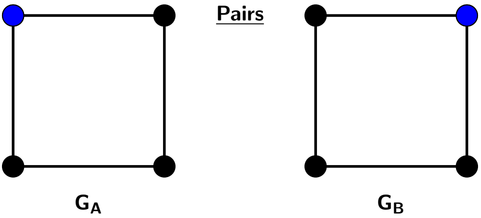

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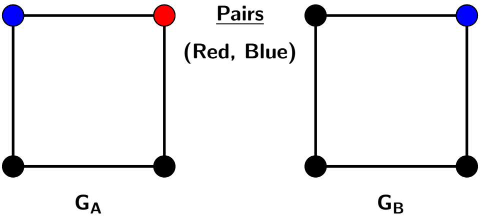

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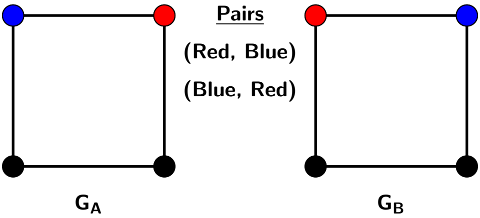

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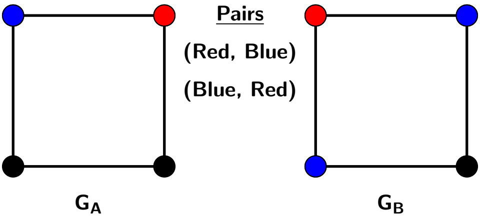

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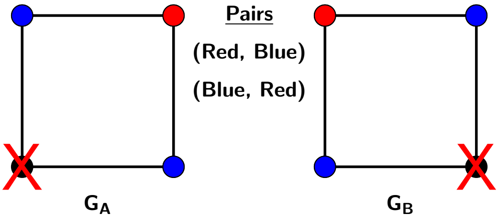
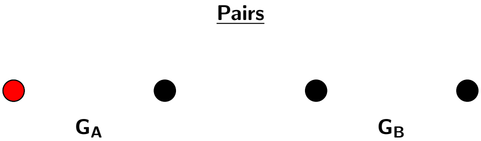


Game Outcomes: Can Alice ever win?

Example: $m = 1$

Case 1:



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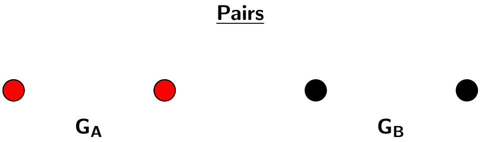

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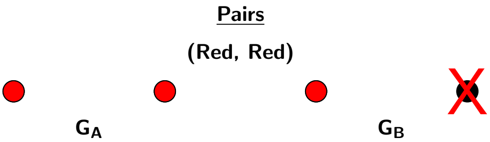
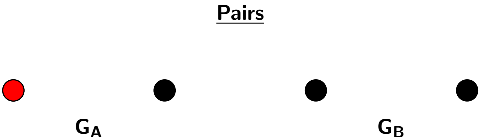


Alice Wins!!

Game Outcomes: Can Alice ever win?

Example: $m = 1$

Case 2:



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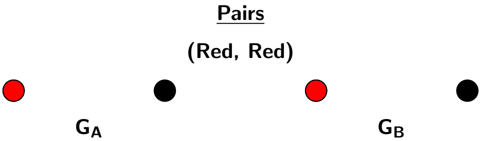
Case 2:



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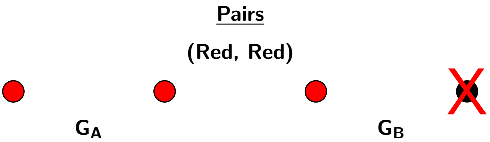
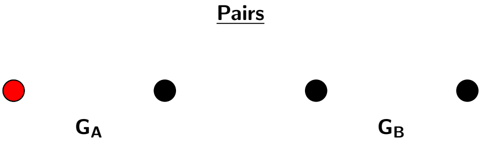


Alice Wins!!

Game Outcomes: Can Alice ever win?

Example: $m = 1$

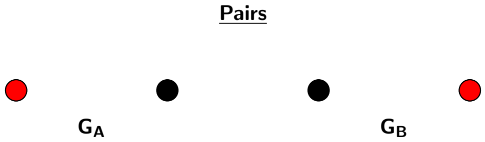
Case 3:



Game Outcomes: Can Alice ever win?

Example: $m = 1$

Case 3:



Game Outcomes: Can Alice ever win?

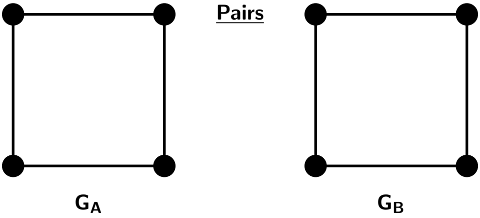
Example: $m = 1$

Case 3:

Alice Wins!!

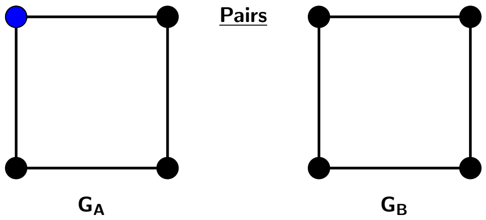
Game Outcomes: Can Bob ever win?

Example: Let $m = 2$.



Game Outcomes: Can Bob ever win?

Example: Let $m = 2$.



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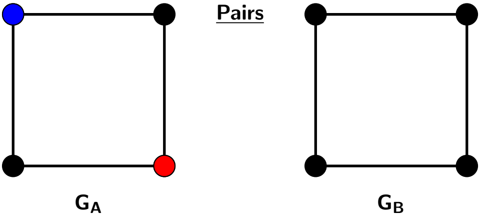

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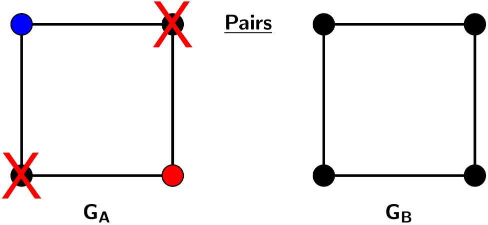

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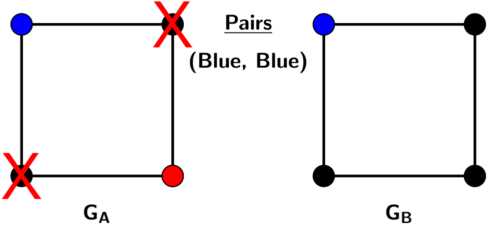

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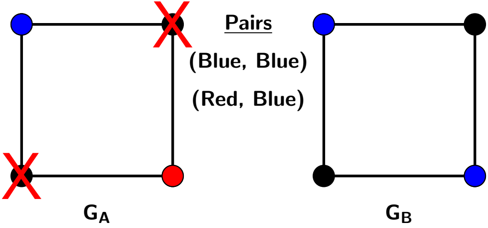

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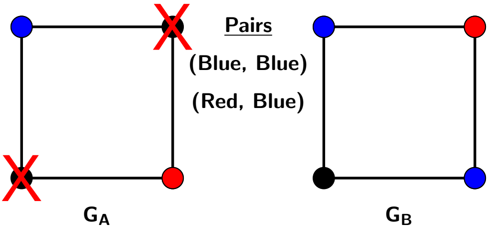

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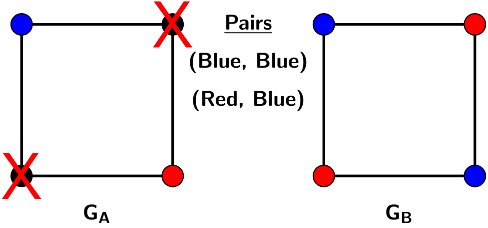

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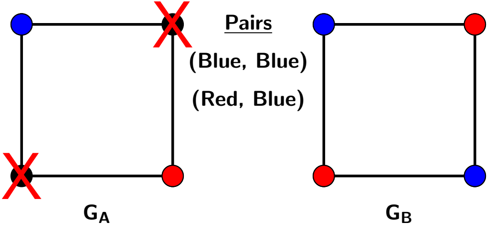

Alice's score: 2, Bob's score: 4. Bob Wins!

What about Draw games?

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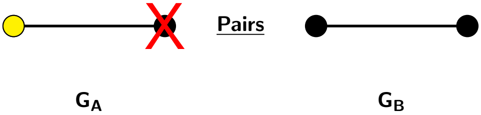

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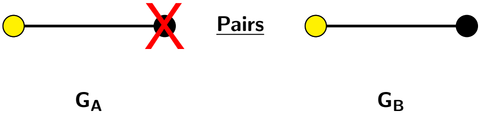

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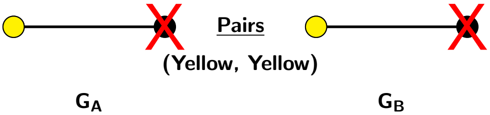


The game is a Draw.

Complexity

Theorem

Determining the outcome of the orthogonal colouring game which includes a partial colouring is PSPACE-complete, for all $m \geq 3$.

Definition

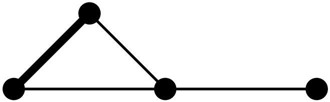
An *involution* of G is an automorphism σ of G with the property

$$\forall v \in V : \quad (\sigma \circ \sigma)(v) = v.$$

We define an involution of G to be *strictly matched* if

(SI 1) the set $F \subseteq V$ of fixed points of σ induces a complete graph, and

(SI 2) for every $v \in V \setminus F$, we have the (matching) edge $v\sigma(v) \in E$.



THE ORTHOGONAL COLOURING GAME: Main Theorem

Theorem

Let G be a graph that admits a strictly matched involution and $m \in \mathbb{N}$. Then the second player has a strategy guaranteeing a draw in the orthogonal colouring game with m colours.

Example: $m = 2$

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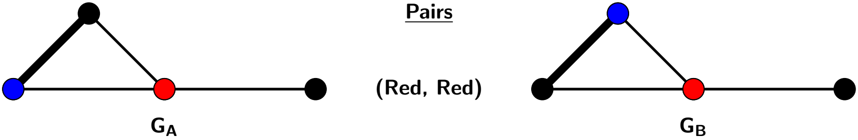

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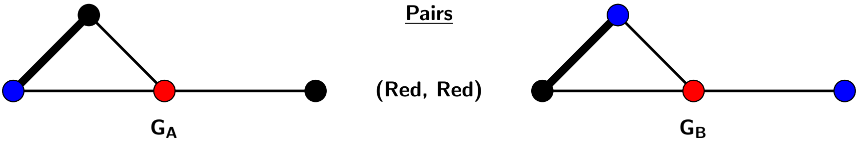

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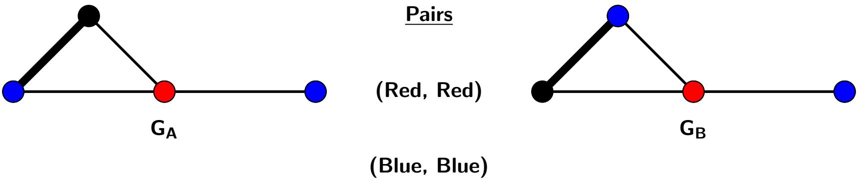

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For any graph G and all $m \in \mathbb{N}$ with $m \geq \Delta(G) + \alpha(G)$, both players have a strategy to draw in the $MOC_m(G)$ game, where $\Delta(G)$ is the max degree of G, $\alpha(G)$ is the stability number of G.

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Lemma

For all $n \in \mathbb{N}$, both players have a strategy to guarantee a draw in the $MOC_1(K_n \square K_n)$ game.

Graph Characterization

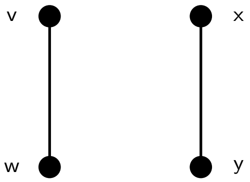
Theorem

A graph G admits a strictly matched involution if and only if its vertex set V can be partitioned into a clique C and a set inducing a graph that has a perfect matching M such that:

1. *for any two edges $vw, xy \in M$, the graph induced by v, w, x, y is isomorphic to*
   - *a $2K_2$ or*
   - *a C_4 or*
   - *a K_4;*
2. *for any edge $vw \in M$ and any vertex $z \in C$, the graph induced by the vertices v, w, z is isomorphic to*
   - *a $K_1 \cup K_2$ or*
   - *a K_3.*

Sketch of proof ('$\Rightarrow$')

Assume G admits a strictly matched involution.
Consider vw, $xy \in M$, and the induced graph of v, w, x, y:



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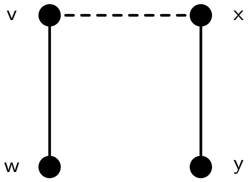

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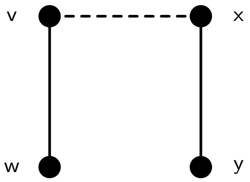

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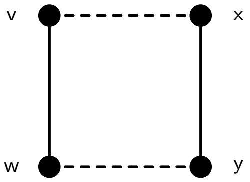

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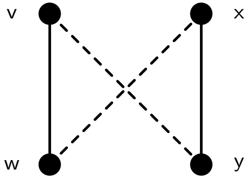

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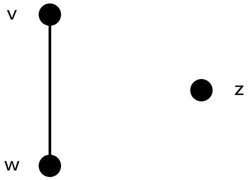

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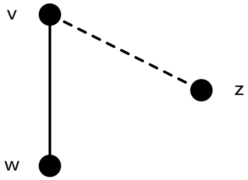

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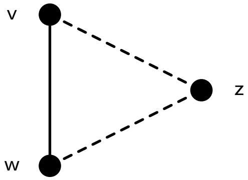
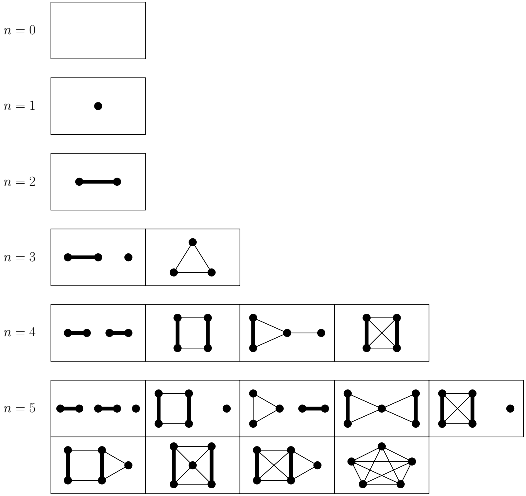


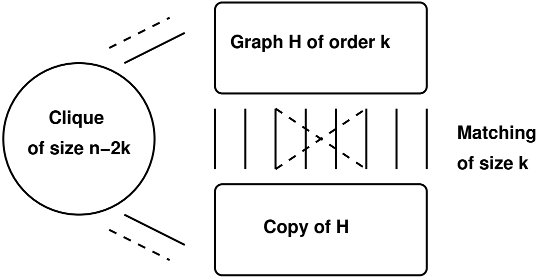
Graphs up to order $n = 5$ which admit a strictly matched involution



Structural results

Corollary

Any graph G on n vertices admitting a strictly matched involution has a partition of its vertex set



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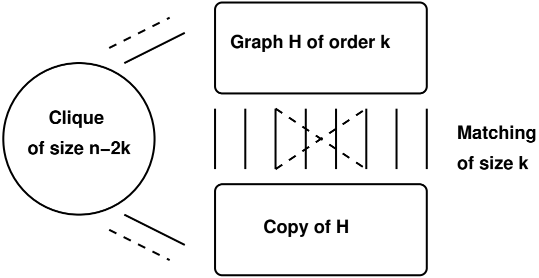

Theorem

Given a graph G, it is NP-complete to determine if G admits a strictly matched involution.

Summary

Summary:

- If G admits a strictly matched involution, then there exists a drawing strategy for Bob.
- Subclass of graphs that also have an Alice drawing strategy (based on the number of colours).
- Characterized graphs that admit a strictly matched involution.

Future Directions

Future Directions:

- Characterize the class of graphs that admit a strictly matched involution for which the game with m colours is a Draw (Bob win resp.).
- Determine the outcome for other classes of graphs.
- What about playing under the misère winning convention?

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Thank you!

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